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Calculus (Tutorial # 6)

Limits, Continuity and Differentiation in $\mathbb{R}^n$

1. If $\mathbf{r}(t) = (t, t^2, t^3)$ then find $\mathbf{r}'(t), T(1), \mathbf{r}''(t)$ and $\mathbf{r}'(t) \times \mathbf{r}''(t)$.

2. Find the point on the curve defined by $\mathbf{r}(t) := (2\cos t, 2\sin t, e^t),\ t \in [0, \pi]$ where tangent is parallel to the plane $\sqrt{3}x + y = 1$.

3. Show that the curves given by $\mathbf{r}_1(t) = (t, t^2, t^3)$ and $\mathbf{r}_2(t) = (\sin t, \sin 2t, t)$ intersect at the origin. Find the angle of their intersection.

4. Let $\mathbf{r}(t)$ be a vector function defined for $t \in (a, b)$. Then

   (a) $\dfrac{d}{dt}[\mathbf{r}(t) \times \mathbf{r}'(t)] = \mathbf{r}(t) \times \mathbf{r}''(t)$.

   (b) If $\mathbf{r}(t) \neq 0$ then $\dfrac{d}{dt}\|\mathbf{r}(t)\| = \dfrac{1}{\|\mathbf{r}(t)\|}\mathbf{r}(t) \cdot \mathbf{r}'(t)$.

   (c) If $\|\mathbf{r}(t)\| = 1$ for each $t \in (a, b)$ then show that $\mathbf{r}(t)$, is normal to the curve defined by $\mathbf{r}$ at each $t \in (a, b)$.

5. Let $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$ be given by $\mathbf{r}(t) = (\cos t, \sin t)$. Prove the following:

   (i) $\mathbf{r}'(t) = (-\sin t, \cos t)$.

   (ii) The equation $\mathbf{r}(2\pi) - \mathbf{r}(0) = \mathbf{r}'(t)(2\pi - 0)$ is not satisfied for any $t \in \mathbb{R}$.

   (iii) For each $\mathbf{a} \in \mathbb{R}^2$, there is $t_0 \in (0, 2\pi)$ such that $\mathbf{a} \cdot (\mathbf{r}(t) - \mathbf{r}(s)) = \mathbf{a} \cdot \mathbf{r}'(t_0)(t - s)$.

6. Sketch the graph of the following surfaces.

   (a) **Sphere:** $S := \{(x, y, z) \in \mathbb{R}^2 :\ x^2 + y^2 + z^2 = 1\}$.

   (b) **Ellipsoid:** $S := \{(x, y, z) \in \mathbb{R}^2 :\ x^2 + \dfrac{y^2}{4} + \dfrac{z^2}{9} = 1\}$.

   (c) **Hyperboloid of one sheet:** $S := \{(x, y, z) \in \mathbb{R}^2 :\ x^2 + y^2 - z^2 = 1\}$.

   (d) **Hyperboloid of two sheets:** $S := \{(x, y, z) \in \mathbb{R}^2 :\ x^2 - y^2 - z^2 = 1\}$

   (e) **Paraboloid :** $S := \{(x, y, z) \in \mathbb{R}^2 :\ z = x^2 + y^2\}$

   (f) **Elliptical paraboloid:** $S := \{(x, y, z) \in \mathbb{R}^2 :\ z = \dfrac{x^2}{4} + \dfrac{y^2}{9}\}$

   (g) **Parabolic cylinder:** $S := \{(x, y, z) \in \mathbb{R}^2 :\ z = x^2\}$

   (h) **Rectangular hyperbolic paraboloid:** $S := \{(x, y, z) \in \mathbb{R}^2 :\ z = x^2 - y^2\}$

   (i) **Hyperbolic paraboloid:** $S := \{(x, y, z) \in \mathbb{R}^2 :\ z = \dfrac{x^2}{4} - \dfrac{y^2}{9}\}$

(j) **Cylinder:** $S := \{(x,y,z) \in \mathbb{R}^3 : \ x^2+y^2=1\}$

(k) **Hyperboloid:** $S := \{(x,y,z) \in \mathbb{R}^3 : \ z = xy\}$

7. Find the following limits if exist

(i) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ (ii) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$ (iii) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y}{x^2+y^4}$

(iv) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^5-xy^4}{x^4+y^4}$ (v) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$ (vi) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^4+y^2}$

(vii) $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x-y^4}{x^3-y^4}$ (viii) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$ (ix) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2}$.

8. Let $D \subset \mathbb{R}^2$ be an open set and $f : D \to \mathbb{R}$ be a function. Then prove the following.

(a) If $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and if one of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ is bounded on D, then f is continuous on D. What about converse?

(b) If both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous on D, then f is differentiable on D. What about the converse?

9. Let $D \subset \mathbb{R}^2$ be an open set and $f : D \to \mathbb{R}$ be a function. If the functions $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are both continuous on D, then $\frac{\partial^2 f}{\partial x \partial y}(a,b) = \frac{\partial^2 f}{\partial y \partial x}(a,b)$ for $(a,b) \in D$.

10. Show that $\displaystyle\lim_{(x,y)\to(2,-2)} e^{\left(\frac{4(x+y)\log(y^2x)}{x^2-y^2}\right)} = 8$.

11. Let $f(x,y) = x^{-1}\sin(xy)$ for $x \neq 0$. How should you define $f(0,y)$ for $y \in \mathbb{R}$ so as to make f a continuous function on all of $\mathbb{R}^2$?

12. At which points of $\mathbb{R}^2$, the following function is continuous?

$$f(x,y) = \begin{cases} \dfrac{y(y-x^2)}{x^4} & \text{if } 0 < y < x^2 \\ 0 & \text{otherwise.} \end{cases}$$

13. Under what conditions on $\alpha, \beta \in \mathbb{R}$, the following function is continuous at $(0,0)$?

$$f(x,y) = \begin{cases} \dfrac{x^\alpha y^\beta}{x^2+xy+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

14. Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(x,a)$ and $f(a,y)$ are continuous functions of x and y respectively, for any $a \in \mathbb{R}$ but f is not continuous on $\mathbb{R}^2$.

15. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(0,0) = 0$ and $f(x,y) = \dfrac{x^2y^2}{x^4+y^2}$ if $(x,y) \neq (0,0)$.

(i) Show that $\dfrac{\partial f}{\partial x}$ is continuous at $(0,0)$, but $\dfrac{\partial f}{\partial y}$ is not.

(ii) Show that f is differentiable at $(0,0)$.

16. Let $f(x,y) = (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right)$ and $f(0,0) = 0$. Show that f is differentiable at $(0,0)$ but both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are discontinuous at $(0,0)$.

17. Let $f : \mathbb{R}^2 \to \mathbb{R}$ is such that $f(0,0) = 0$, and is given by following for $(x,y) \neq (0,0)$.

(i) $f(x,y) = \dfrac{xy}{x^2+y^2}$ (ii) $f(x,y) = \sqrt{|xy|}$ (iii) $f(x,y) = \dfrac{x^2y^2}{x^2y^2+(y-x)^2}$

(iv) $f(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}}$ (v) $f(x,y) = \dfrac{x^2y}{x^2+y^2}$ (vi) $f(x,y) = |x| + |y|$

Answer the following for each of the functions defined above.

(a) For which vectors $u \in \mathbb{R}^2 \setminus \{(0,0\}$ does $D_u f(0,0)$ exist? Evaluate it when it exists.

(b) Do $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$ exist?

(c) Is f continuous at $(0,0)$?

(d) Is f differentiable at $(0,0)$?

18. Let $f(x,y) = |xy|$ for $(x,y) \in \mathbb{R}^2$. Prove the following:

(i) f_x exists at all points of $\mathbb{R}^2$ except at $(0,b)$ for $b \in \mathbb{R} \setminus \{0\}$. (Do the same for f_y.)

(ii) f is differentiable at $(0,0)$.

19. Let $f(x,y) = \dfrac{x^2y}{x^2+y^2}$ for $(x,y) \neq (0,0)$ and $f(0,0) = 0$.

(i) Show that for any non-zero vector (u,v), $D_u f(0,0) = u^2v$.

(ii) Is f differentiable at $(0,0)$?

20. Find the indicated partial derivatives.

(a) $f(x,y) = \ln(x + \sqrt{x^2+y^2})$; $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at $(x,y) = (3,4)$.

(b) If $z = f(x+y)$ then find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

(c) If $z = f(xy)$ then find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

(d) If $f(x,y) = x^4y^2 - x^3y$; then find all the partial derivatives of order 3.

21. Verify that the function $z = \ln(e^x + e^y)$ is a solution of the differential equations

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1 \qquad \text{and} \qquad \frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 0.$$

22. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(0,0) = 0$ and $f(x,y) = \dfrac{x^3y - xy^3}{x^2+y^2}$ if $(x,y) \neq (0,0)$.

    (a) Show that $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ exist for all $(x,y) \in \mathbb{R}^2$. Also find the expressions for them.

    (b) Show that $\dfrac{\partial^2 f}{\partial x \partial y}(0,0) = -1$ and $\dfrac{\partial^2 f}{\partial y \partial x}(0,0) = 1$.

23. Let $D \subset \mathbb{R}^2$ be an open set and suppose the function $f : D \to \mathbb{R}$ has a local maximum at $(x_0, y_0) \in D$. If $D_u f(x_0, y_0)$ exists for all $u \in \mathbb{R}^2 \setminus \{(0,0)\}$ then $D_u f(x_0, y_0) = 0$. In particular $\dfrac{\partial f}{\partial x}(x_0, y_0) = 0$ and $\frac{\partial f}{\partial y}(x_0, y_0) = 0$.

24. Find the point on the plane $x - 2y + 3z = 6$ that is closest to the point $(0,1,1)$.

25. Find the points on the cone $z^2 = x^2 + y^2$ that are closest to the point $(4,2,0)$.

26. Find three positive numbers whose sum is 100 and product is maximum.

27. Find the absolute maximum and minimum values of f on the set D.

    (a) $f(x,y) = x^2 + y^2 - 2x$, D is triangular region with vertices $(2,0)$ $(0,2)$ and $(0,-2)$.

    (b) $f(x,y) = x^4 + y^4 - 4xy + 2$, $D := \{(x,y) : \ 0 \leq x \leq 3, \ 0 \leq y \leq 2\}$.

    (c) $f(x,y) = x^2 = y^2 + x^2y + 4$, $D := \{(x,y) : \ |x| \leq 1, \ |y| \leq 1\}$.

    (d) $f(x,y) = 2x^3 + y^4 \quad D := \{(x,y) : \ x^2 + y^2 \leq 1\}$.

28. Find the local maximum and minimum for the following functions.

    (a) $f(x,y) = 9 - 2x + 4y - x^2 - 4y^2$

    (b) $f(x,y) = xy(1 - x - y)$

    (c) $f(x,y) = xe^{-2x^2 - 2y^2}$

    (d) $f(x,y) = \sin x \sin y$, $-\pi < x < \pi$ and $-\pi < y < \pi$.

29. If a function of one variable is differentiable on an interval and has only one critical point, then a local maximum has to be an absolute or global maximum. But this is not true for function of two variables. Show that the function $f(x,y) = 3xe^y - x^3 - e^{3y}$ has exactly one critical point, and that f has a local maximum there but that is not an absolute maximum.

30. Show that the function $f(x, y) = x^2ye^{-x^2-y^2}$ has maximum values at $\left(\pm 1, 1/\sqrt{2}\right)$ and minimum values at $\left(\pm 1, -1/\sqrt{2}\right)$. Show also that f has infinitely many other critical points and $D := \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$ is zero at each of them. Which of them give rise to maximum values? Minimum values? Saddle points?

31. Let $g : (0, \infty) \to \mathbb{R}$ be a two times continuously differentiable function. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) := g(r)$ where $r = \sqrt{x^2 + y^2}$. Then using the polar coordinates and chain rule show that
$$\Delta f := \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{d^2 g}{dr^2} + \frac{2}{r}\frac{dg}{dr}.$$
Hence, solve $\Delta f = 0$ in $\mathbb{R}^2$.

32. If $u(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$, for $(x, y, z) \neq (0, 0, 0)$ then find ∇u and $\Delta u := \nabla \cdot (\nabla u)$.

33. A function $f : \mathbb{R}^2 \to \mathbb{R}$ is said to be **homogeneous** of degree $\alpha \in \mathbb{R}$ if $f(tX) = t^\alpha f(X)$ for all $X := (x, y) \in \mathbb{R}^2$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a two times differentiable and homogeneous function of degree α. Then show that

    (a) (Euler's Theorem) $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = \alpha f(x, y)$

    (b) $x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = \alpha(\alpha - 1)f(x, y)$.

34. Let $f, g : \mathbb{R}^3 \to \mathbb{R}$ and $F, G : \mathbb{R}^3 \to \mathbb{R}^3$ be differentiable functions. Then prove the following.

    (a) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

    (b) $\nabla(F \cdot G) = (F \cdot \nabla)G + F \times (\nabla \times G) + (G \cdot \nabla)F + G \times (\nabla \times F)$

    (c) $\nabla \cdot (fG) = (\nabla f) \cdot G + f(\nabla \cdot G)$

    (d) $\nabla \cdot (F \times G) = G \cdot (\nabla \times F) - F \cdot (\nabla \times G)$

    (e) $\nabla \times (fG) = f(\nabla \times G) + (\nabla f) \cdot G$

    (f) $\nabla \times (F \times G) = (G \cdot \nabla)F + (\nabla \cdot G)F - (F \cdot \nabla)G - (\nabla \cdot F)G$

35. Let $f : \mathbb{R}^3 \to \mathbb{R}$ and $F : \mathbb{R}^3 \to \mathbb{R}^3$ be two times differentiable functions. Then prove the following.

    (a) $\nabla \times (\nabla f) = 0$

    (b) $\nabla \cdot (\nabla \times F) = 0$

    (c) $\nabla \times (\nabla \times F) = \nabla(\nabla \cdot F) - \Delta F$, here $(\Delta F)_i := \frac{\partial^2 F_i}{\partial x^2} + \frac{\partial^2 F_i}{\partial y^2} + \frac{\partial^2 F_i}{\partial z^2}$, is the i′th component of ΔF, for $i = 1, 2, 3$.

Optional problems

1. Find the interior and boundary points for the following sets. Tell which of them is open, closed or neither.

   (i) $[1,2]$, $(2,3)$ and $[3,7)$

   (ii) $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}\setminus\mathbb{Q}$ and $\mathbb{R}$

   (iii) $\{(x,y)\in\mathbb{R}:\ 0<x^2+y^2\leq 16\}$

   (iv) $\{(x,y)\in\mathbb{R}:\ 0<x^2+y^2\leq 16\}$

   (v) $\{(x,y): |x|+|y|\leq 1\}$

   (vi) $\{(x,y)\in\mathbb{R}^2: x^2-x\leq y\leq 0\}$

   (vii) $\{(x,y)\in\mathbb{R}^2:\ x,y\in\mathbb{Q}\cap[0,1]\}$

   (viii) $\mathbb{R}^2\setminus\{0\}$

2. Answer the following with justification.

   (a) True or false: If S_1 and S_2 are open subset of $\mathbb{R}^n$ then $S_1\cap S_2$ is also open.

   (b) True or false: If $\{S_j\}_{j\in\mathbb{N}}$ is a collection of open subsets of $\mathbb{R}^n$ then so is $\cap_{j\in\mathbb{N}}S_j$.

   (c) True or false: If $\{S_j\}_{j\in\mathbb{N}}$ is a collection of closed subsets of $\mathbb{R}^n$ then $\cup_{j\in\mathbb{N}}S_j$ may not be closed.

   (d) If $\{S_j\}_{j\in\mathbb{N}}$ is a collection of open subsets of $\mathbb{R}^n$ then so is $\cup_{j\in\mathbb{N}}S_j$.

   (e) True or false: Intersection of finite number of open sets is open.

   (f) True or false: Union of finite number of closed sets is closed.

3. Let $S\subset\mathbb{R}^n$ be a closed set and $\{X_k\}_{k\in\mathbb{N}}$ be a sequence in S such that $\{X_k\}_{k\in\mathbb{N}}$ converges to $X\in\mathbb{R}^n$, then show that $X\in S$. What would be your conclusion if S is not closed?

4. Let $S\subset\mathbb{R}^n$ be a nonempty set and $A\in\partial S$. Then show that there exists a sequence $\{X_k\}_{k\in\mathbb{N}}\subseteq S$ such that $X_k\to A$ as $k\to\infty$.

5. Prove that a function $F:\mathbb{R}^n\to\mathbb{R}^m$ is a linear map if and only if there exists an $m\times n$ matrix A such that $F(X)=AX$ for any column vector $X\in\mathbb{R}^n$.

6. Let $f: D\subset\mathbb{R}\to\mathbb{R}$ be a continuous function. Then show that the graph $G_f := \{x,y)\in\mathbb{R}^2:\ y=f(x) \text{ and } x\in D\}$ and the level set $L_f(c):=\{x\in D:\ f(x)=c\}$ are closed subset of $\mathbb{R}^2$ and $\mathbb{R}$ respectively.