Department of Mathematics, IIT Jammu, India
Instructors: Rahul Kitture and Manmohan Vashisth.

Marks: 45
Time: 3:00 PM to 5:00 PM

Calculus (SMD001UM1) End-sem exam

- Attempt any 9 questions.
- Justify each step clearly.
- You can refer to the Tutorial problems, class notes or book (write name with page number), if you are using it. Otherwise, it will not be considered.

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1. Let $\{a_n\}$ be a sequence of real numbers such that $|a_{n+1} - a_n| < \frac{7}{8}|a_{n-1} - a_n|$. Prove that $\{a_n\}$ is convergent (give details). **(5 Marks)**

2. Discuss the convergence of the following series.

   (a) $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n^{100}}\right)$. **(2.5 Marks)**

   (b) $\sum_{n=1}^{\infty} \frac{n!}{n^n 2^n}$. **(2.5 Marks)**

3. For any integer $n > 2$, show that $f(x) = x^n - ax - b$ can have at most three distinct roots in $\mathbb{R}$. **(5 Marks)**

4. Determine the points $a \in \mathbb{R}$ at which the functions $f(x) = |\cos(x)|$ and $g(x) = \cos(|x|)$ are differentiable (write details). **(5 Marks)**

5. (a) Let $f : [-1, 1] \to \mathbb{R}$ be a continuous and let

   $$a_n := \int_{-1}^{1} t^n \, f(t) \, dt, \qquad \text{for } n \in \mathbb{N}.$$

   Check the convergence of the sequence $\{a_n\}$. **(3 Marks)**

   (b) Let

   $$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ -1, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

   Then show that f is not integrable on any interval in $\mathbb{R}$. **(2 Marks)**

6. Let $f : [0,1] \to [0,\infty)$ be a continuous function. Then show that $\displaystyle\int_0^1 f(x)\, dx > 0$ if and only if there exists at least one point $x_0 \in [0,1]$ such that $f(x_0) > 0$. **(5 Marks)**

7. Define $f : [1,\infty) \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & \text{if } x \in \left[n, n+\frac{1}{2^n}\right] \text{ for } n \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases}$$

Then show that the improper integral $\displaystyle\int_1^\infty f(x)\, dx$ converges to 1 but the series $\sum_{n=1}^{\infty} f(n)$ is divergent. **(5 Marks)**

8. Determine if the following function is continuous at $(0,0)$. **(5 Marks)**

$$f(x,y) = \begin{cases} \dfrac{x\sin(x)\sin(y)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{otherwise.} \end{cases}$$

9. Let $f(x,y) = \dfrac{|xy|}{\sqrt{x^2+y^2+1}}$.

   (a) Does $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$ exist? **(2.5 Marks)**

   (b) Is f differentiable at $(0,0)$? **(2.5 Marks)**

10. Let $f, g : \mathbb{R} \to \mathbb{R}$ be two times-differentiable functions, and let $u : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$u(x,t) := f(x+t) + g(x-t).$$

Then show that

$$\frac{\partial^2 u}{\partial t^2}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = 0 \qquad \text{for } (x,t) \in \mathbb{R}^2$$

(5 Marks)

11. Find the point on the line passing through $(1,0,0)$ and $(0,1,0)$ that is closest to the line through $(0,0,0)$ and $(1,1,1)$. **(5 Marks)**

12. Find the extreme values of the function $f(x,y) = x^3 - x + y^2 - 2y$ on the closed triangular region with vertices at $(-1,0)$, $(1,0)$ and $(0,2)$. **(5 Marks)**