# A PARTIAL DATA INVERSE PROBLEM FOR THE CONVECTION-DIFFUSION EQUATION

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Abstract. In this article we study the inverse problem of determining the convection term and the time-dependent density coefficient appearing in the convection-diffusion equation. We prove the unique determination of these coefficients from the knowledge of solution measured on a subset of the boundary.

**Keywords:** Inverse problems, parabolic equation, Carleman estimates, partial boundary data.

**Mathematics subject classification 2010:** 35R30, 35K20.

## 1. Introduction

Let $\Omega \subset \mathbb{R}^n$ with $n \geq 2$, be a bounded simply connected open set with $C^2$ boundary. For $T > 0$, let $Q := (0,T) \times \Omega$ and denote its lateral boundary by $\Sigma := (0,T) \times \partial\Omega$. We consider the following initial boundary value problem

$$\begin{cases} \left(\partial_t - \sum_{j=1}^n (\partial_j + A_j(t,x))^2 + q(t,x)\right) u(t,x) = 0, \ (t,x) \in Q \\ u(0,x) = 0, \ x \in \Omega \\ u(t,x) = f(t,x), \ (t,x) \in \Sigma. \end{cases} \tag{1.1}$$

Throughout this article, we assume that $A_j \in W^{1,\infty}(Q)$ for $1 \leq j \leq n$ and $q \in L^\infty(Q)$. Let us denote by

$$A(t,x) := (A_1(t,x), A_2(t,x), \cdots, A_n(t,x))$$

and by

$$\mathcal{L}_{A,q} := \partial_t - \sum_{j=1}^n (\partial_j + A_j(t,x))^2 + q(t,x).$$

Before going to the main context of the article, let us briefly mention about the well-posedness of the forward problem. Following [14], define the spaces $\mathcal{K}_0$ and $\mathcal{H}_T$ by

$$\mathcal{K}_0 := \left\{ f|_\Sigma : \ f \in L^2\left(0,T; H^1(\Omega)\right) \cap H^1\left(0,T; H^{-1}(\Omega)\right) \text{ and } f(0,x) = 0, \text{ for } x \in \Omega \right\}$$
$$\text{and } \mathcal{H}_T := \left\{ g|_\Sigma : \ g \in H^1\left(0,T; H^1(\Omega)\right) \text{ and } g(T,x) = 0, \text{ for } x \in \Omega \right\}.$$

As shown in [14] (see also [45]) that for $f \in \mathcal{K}_0$, Equation (1.1) admits a unique solution $u \in H^1(0,T; H^{-1}(\Omega)) \cap L^2(0,T; H^1(\Omega))$ and the operator $\mathcal{N}_{A,q}u$ given by

$$\langle \mathcal{N}_{A,q}u, w|_\Sigma \rangle := \int_Q \left( -u\partial_t \overline{w} + \nabla_x u \cdot \nabla_x \overline{w} + 2uA \cdot \nabla_x \overline{w} + (\nabla_x \cdot A)u\overline{w} - |A|^2 u\overline{w} + qu\overline{w} \right) \mathrm{d}x\mathrm{d}t$$

is well-defined for $w \in H^1(Q)$ such that $w(T,x) = 0$, for $x \in \Omega$. Note that if $A, q$ and $f$ are smooth enough then $\mathcal{N}_{A,q}u$ is given by

$$\mathcal{N}_{A,q}u = (\partial_\nu u + 2\,(\nu \cdot A)\,u)\,|_\Sigma$$

where $\nu$ stands for the outward unit normal vector to $\partial\Omega$ and $u$ solution to (1.1). Motivated by this and [14], we define the Dirichlet to Neumann (DN) map $\Lambda_{A,q} : \mathcal{K}_0 \to \mathcal{H}_T^*$ by

$$\Lambda_{A,q}(f) := \mathcal{N}_{A,q}u \tag{1.2}$$

where $\mathcal{H}_T^*$ denotes the dual of space $\mathcal{H}_T$ and $u$ is solution to (1.1) with Dirichlet boundary data equal to $f$. Then from ([14], see Section 2), we have that DN map $\Lambda_{A,q}$ defined by (1.2) is continuous from $\mathcal{K}_0$ to $\mathcal{H}_T^*$.

In the present article we first consider the problem of unique recovery of coefficients $A(x)$ and $q(t,x)$ appearing in (1.1) from the information of DN map $\Lambda_{A,q}$ measured on a subset of $\Sigma$. It is well-known [see [53]] that one cannot determine coefficient $A(x)$ uniquely from DN map $\Lambda_{A,q}$ measured on $\Sigma$ and this is because of the gauge invariance associated with $A(x)$. So one can only hope to recover $A(x)$ uniquely upto a potential term however the coefficient $q(t,x)$ can be determined uniquely (see Theorem 2.1 in §2 for more details). Later as a corollary of Theorem 2.1, we consider the problem of determining time-dependent coefficients $A(t,x)$ and $q(t,x)$ appearing in (1.1) from the partial information of DN map $\Lambda_{A,q}$. Using some extra assumption on $A(t,x)$ and Theorem 2.1, we show that time-dependent coefficients $A(t,x)$ and $q(t,x)$ can be determined uniquely from the knowledge of DN map $\Lambda_{A,q}$ measured on a part of $\Sigma$ (see Corollary 2.2 below in §2 for more details).

The initial boundary value problem (1.1) is known as a convection-diffusion equation with constant diffusion. The coefficients $A$ and $q$ are called convection term and density coefficient respectively. The convection-diffusion equations appear in chemical engineering, heat transfer and probabilistic study of diffusion process etc.

Determination of the coefficients from boundary measurements appearing in parabolic partial differential equations have been studied by several authors. Isakov in [33] considered the problem of determining time-independent coefficient $q$ for the case when $A = 0$ in (1.1) from the DN map and he proved the uniqueness result by showing the density of product of solutions (inspired by the work of [54]) in some Lebesgue space. Avdonin and Seidman in [2] studied the problem of determining time-independent density coefficient $q(x)$ appearing in (1.1) by using the boundary control method pioneered by Belishev, Kurylev, Lassas and others see [1, 4, 37] and references therein. In [23] Choulli proved the stability estimate analogous to the uniqueness problem considered in [33]. In [25] problem of determining the first order coefficients appearing in a parabolic equations in one dimension from the data measured at final time is studied. Cheng and Yamamoto in [16] proved the unique determination of convection term $A(x)$ (when $q = 0$ in (1.1)) from a single boundary measurement in two dimension. Gaitan and Kian [30] using the global Carleman estimate used for hyperbolic equations [see [12]] proved the stable determination of time-dependent coefficient $q(t,x)$ in a bounded waveguide. Choulli and Kian in [21] proved the stability estimate for determining time-dependent coefficient $q$ from the partial DN map. For more works related to parabolic inverse problems, we refer to [8, 18, 19, 20, 22, 23, 25, 30, 34, 35, 46] and the references therein. We also mention the work of [3, 5, 9, 11, 27, 41, 42, 43] related to dynamical Schrödinger equation and the work of [10, 28, 29, 49, 50, 51] for hyperbolic inverse problems. We refer to [15, 17, 47] for steady state convection-diffusion equation. Recently Caro and Kian in [14] established the unique determination of convection coefficient together with non-linearity term appearing in the equation from the knowledge of DN map measured on $\Sigma$.

Inspired by the work of [21], we consider the problem of determining the full first order space derivative perturbation of heat operator from the partial DN map. We have proved our uniqueness result by using the geometric optics solutions constructed using a Carleman estimate in a Sobolev space of negative order and inverting the ray transform of a vector field which is known only in a very small neighbourhood of fixed direction $\omega_0 \in \mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\}$. For elliptic and hyperbolic inverse problems these kind of techniques have been used by several authors. Related to our work, we refer to [13, 26] for the elliptic case and to [6, 7, 32, 36, 38, 39, 40, 44] for the hyperbolic case.

The article is organized as follows. In §2 we give the statement of the main result. §3 contains the boundary Carleman estimate. In §4 we construct the geometric optics solutions using a Carleman estimate in a Sobolev space of negative order. In §5 we derive an integral identity and §6 contains the proof of main Theorem 2.1 and Corollary 2.2.

## 2. Statement of the main result

We begin this section by fixing some notation which will be used to state the main result of this article. Following [13] fix an $\omega_0 \in \mathbb{S}^{n-1}$ and define the $\omega_0$-shadowed and $\omega_0$-illuminated faces by

$$\partial\Omega_{+,\omega_0} := \{x \in \partial\Omega : \ \nu(x)\cdot\omega_0 \geq 0\}, \quad \partial\Omega_{-,\omega_0} := \{x \in \partial\Omega : \ \nu(x)\cdot\omega_0 \leq 0\}$$

of $\partial\Omega$ where $\nu(x)$ is outward unit normal to $\partial\Omega$ at $x \in \partial\Omega$. Corresponding to $\partial\Omega_{\pm,\omega_0}$, we denote the lateral boundary parts by $\Sigma_{\pm,\omega_0} := (0,T)\times\partial\Omega_{\pm,\omega_0}$. We denote by $F = (0,T)\times F'$ and $G = (0,T)\times G'$ where $F'$ and $G'$ are small enough open neighbourhoods of $\partial\Omega_{+,\omega_0}$ and $\partial\Omega_{-,\omega_0}$ respectively in $\partial\Omega$.

Since $\Omega$ is bounded and $T < \infty$, so we can choose a smallest $R > 0$ such that $\overline{Q} \subset B(0,R)$ where $B(0,R) \subset \mathbb{R}^{1+n}$ is a ball of radius $R$ with center at origin. Now we define admissible set $\mathcal{A}$ of vector fields $A(t,x)$ appearing in (1.1) by

$$\mathcal{A} := \left\{A \in W^{1,\infty}(Q) : \ \|A\|_\infty \leq \frac{1}{9R}\right\}. \tag{2.1}$$

We first prove the uniqueness result for time-independent convection coefficient $A \in \mathcal{A}$ and time-dependent density coefficient $q$. More precisely we prove the following theorem:

**Theorem 2.1.** *Let $\left(A^{(1)}, q_1\right)$ and $\left(A^{(2)}, q_2\right)$ be two sets of coefficients such that $A^{(i)} \in \mathcal{A}$ are time-independent and $q_i \in L^\infty(Q)$ for $i = 1,2$. Let $u_i$ be the solutions to (1.1) when $(A,q) = \left(A^{(i)}, q_i\right)$ and $\Lambda_{A^{(i)},q_i}$ for $i = 1,2$ be the DN maps defined by (1.2) corresponding to $u_i$. Now if*

$$\Lambda_{A^{(1)},q_1}(f)|_G = \Lambda_{A^{(2)},q_2}(f)|_G, \ \text{for } f \in L^2\left(0,T; H^{1/2}(\partial\Omega)\right) \tag{2.2}$$

*then there exists a function $\Phi \in W_0^{2,\infty}(\Omega)$ such that*

$$A^{(1)}(x) - A^{(2)}(x) = \nabla_x\Phi(x), \quad x \in \Omega$$

*and*

$$q_1(t,x) = q_2(t,x), \ (t,x) \in Q$$

*provided $A^{(1)}(x) = A^{(2)}(x)$ for $x \in \partial\Omega$.*

In Theorem 2.1 if we take some extra assumption on convection term $A^{(i)}$ then we can prove the uniqueness result for full recovery of $A^{(i)}$ even for the case when $A^{(i)} \in \mathcal{A}$ for $i = 1,2$ are time-dependent. The precise statement of this is given in the following Corollary.

**Corollary 2.2.** *Let $\left(A^{(1)}, q_1\right)$ and $\left(A^{(2)}, q_2\right)$ be two sets of time-dependent coefficients such that $A^{(i)} \in \mathcal{A}$ and $q_i \in L^\infty(Q)$ for $i = 1, 2$. Let $u_i$ be the solutions to* (1.1) *when $(A, q) = \left(A^{(i)}, q_i\right)$ and $\Lambda_{A^{(i)},q_i}$ for $i = 1, 2$ be the DN maps defined by* (1.2) *corresponding to $u_i$. Now if*

$$\nabla_x \cdot A^{(1)}(t, x) = \nabla_x \cdot A^{(2)}(t, x), \ (t, x) \in Q \tag{2.3}$$

*and*

$$\Lambda_{A^{(1)},q_1}(f)|_G = \Lambda_{A^{(2)},q_2}(f)|_G, \ f \in L^2\left(0, T; H^{1/2}(\partial\Omega)\right)$$

*then we have*

$$A^{(1)}(t, x) = A^{(2)}(t, x) \text{ and } q_1(t, x) = q_2(t, x), \ (t, x) \in Q$$

*provided $A^{(1)}(t, x) = A^{(2)}(t, x)$ for $(t, x) \in \Sigma$.*

**Remark 2.3.** *The additional assumption* (2.3) *on convection term $A^{(i)}$ in Corollary 2.2 have been considered in prior works as well. See for example* [9, 24] *for the determination of vector field term appearing in the dynamical Schrödinger equation and also in* [14] *for non-linear parabolic equation.*

## 3. Boundary Carleman estimate

In this section we prove a Carleman estimate involving the boundary terms for the operator $\mathcal{L}_{A,q}$. We will use this estimate to control the boundary terms appearing in integral identity given by (5.9) where no information is given.

**Theorem 3.1.** *Let $\varphi(t, x) = \lambda^2 t + \lambda\omega \cdot x$ where $\omega \in \mathbb{S}^{n-1}$ is fixed. Let $u \in C^2(\overline{Q})$ such that*

$$u(0, x) = 0, \text{ for } x \in \Omega \text{ and } u(t, x) = 0, \text{ for } (t, x) \in \Sigma.$$

*If $A \in \mathcal{A}$ and $q \in L^\infty(Q)$ then there exists $C > 0$ depending only on $\Omega, T, q$ and $A$ such that*

$$\begin{aligned}\lambda^2 \int_Q e^{-2\varphi}|u(t, x)|^2 \mathrm{d}x\mathrm{d}t + \int_Q e^{-2\varphi}|\nabla_x u(t, x)|^2 \mathrm{d}x\mathrm{d}t + \int_\Omega e^{-2\varphi(T,x)}|u(T, x)|^2 \mathrm{d}x \\ + \lambda \int_{\Sigma_{+,\omega}} e^{-2\varphi}|\partial_\nu u(t, x)|^2 |\omega \cdot \nu(x)| \mathrm{d}S_x \mathrm{d}t \leq C \int_Q e^{-2\varphi}|\mathcal{L}_{A,q} u(t, x)|^2 \mathrm{d}x\mathrm{d}t \\ + C\lambda \int_{\Sigma_{-,\omega}} e^{-2\varphi}|\partial_\nu u(t, x)|^2 \ |\omega \cdot \nu(x)| \mathrm{d}S_x \mathrm{d}t\end{aligned} \tag{3.1}$$

*holds for $\lambda$ large.*

*Proof.* Let

$$\mathcal{L}_\varphi := e^{-\varphi}\mathcal{L}_{A,q} e^\varphi \tag{3.2}$$

and denote by

$$\widetilde{q}(t, x) := q(t, x) - \nabla_x \cdot A(t, x) - |A(t, x)|^2.$$

Then

$$\begin{aligned}\mathcal{L}_\varphi v(t, x) &= e^{-\varphi}\left(\partial_t - \Delta - 2A(t, x) \cdot \nabla_x + \widetilde{q}(t, x)\right)\left(e^\varphi v(t, x)\right) \\ &= \left(\partial_t - \Delta - 2\nabla_x\varphi \cdot \nabla_x\right) v(t, x) + \left(\partial_t\varphi - |\nabla_x\varphi|^2 - \Delta\varphi\right) v(t, x) \\ &\quad - 2\left(A(t, x) \cdot \nabla_x - 2A(t, x) \cdot \nabla_x\phi\right) v(t, x) := \left(P_1 v + P_2 v + P_3 v\right)(t, x)\end{aligned} \tag{3.3}$$

where

$$\begin{aligned} &P_1 := -\Delta + \partial_t\varphi - |\nabla\varphi|^2 - \Delta\varphi = -\Delta \\ &P_2 := \partial_t - 2\nabla_x\varphi\cdot\nabla_x = \partial_t - 2\lambda\omega\cdot\nabla_x \\ &P_3 := -2A(t,x)\cdot\nabla_x - 2A(t,x)\cdot\nabla_x\varphi + \widetilde{q}(t,x) = -2A(t,x)\cdot\nabla_x - 2\lambda\omega\cdot A(t,x) + \widetilde{q}(t,x). \end{aligned} \tag{3.4}$$

Now let

$$I := \int\limits_Q |\mathcal{L}_\varphi v(t,x)|^2 \mathrm{d}x\mathrm{d}t \geq \frac{1}{2}\int\limits_Q |(P_1+P_2)\,v(t,x)|^2\mathrm{d}x\mathrm{d}t - \int\limits_Q |P_3 v(t,x)|^2\mathrm{d}x\mathrm{d}t := I_1 - I_2 \tag{3.5}$$

where

$$I_1 := \frac{1}{2}\int\limits_Q |(P_1+P_2)\,v(t,x)|^2\mathrm{d}x\mathrm{d}t \text{ and } I_2 := \int\limits_Q |P_3 v(t,x)|^2\mathrm{d}x\mathrm{d}t.$$

Next we estimate each of $I_j$ for $j=1,2$. Now $I_1$ is

$$I_1 = \frac{1}{2}\int\limits_Q |P_1 v(t,x)|^2 + \frac{1}{2}\int_Q |P_2 v(t,x)|^2\mathrm{d}x\mathrm{d}t + \int\limits_Q P_1 v(t,x)\; P_2 v(t,x)\mathrm{d}x\mathrm{d}t.$$

We consider each term separately on right hand side of the above equation. Using integration by parts and the fact that $v|_\Sigma = 0$, we have

$$\int\limits_Q |\nabla_x v(t,x)|^2\mathrm{d}x\mathrm{d}t = -\int\limits_Q v(t,x)\Delta v(t,x)\mathrm{d}x\mathrm{d}t \leq \frac{1}{2s}\int\limits_Q |\Delta v(t,x)|^2\mathrm{d}x\mathrm{d}t + \frac{s}{2}\int_Q |v(t,x)|^2\mathrm{d}x\mathrm{d}t$$

holds for any $s>0$. Thus we have

$$\int\limits_Q |P_1 v(t,x)|^2\mathrm{d}x\mathrm{d}t \geq 2s\int\limits_Q |\nabla v(t,x)|^2\mathrm{d}x\mathrm{d}t - s^2\int\limits_Q |v(t,x)|^2\mathrm{d}x\mathrm{d}t. \tag{3.6}$$

Following the proof of [21, Lemma 3.1] we have

$$\int\limits_Q |P_2 v(t,x)|^2\mathrm{d}x\mathrm{d}t \geq \frac{1+4\lambda^2}{16R^2}\int\limits_Q |v(t,x)|^2\mathrm{d}x\mathrm{d}t \tag{3.7}$$

where $R>0$ is the radius of smallest ball $B(0,R)\subset\mathbb{R}^{1+n}$ such that $\overline{Q}\subseteq B(0,R)$. Now consider

$$\begin{aligned} 2\int_Q P_1 v(t,x)\; P_2 v(t,x)\mathrm{d}x\mathrm{d}t &= -2\int\limits_Q \Delta v(t,x)\partial_t v(t,x)\mathrm{d}x\mathrm{d}t + 2\lambda\int\limits_Q \Delta v(t,x)\omega\cdot\nabla_x v(t,x)\mathrm{d}x\mathrm{d}t \\ &= \int\limits_\Omega |\nabla v(T,x)|^2\mathrm{d}x + \lambda\int\limits_\Sigma \omega\cdot\nu(x)|\partial_\nu v(t,x)|^2\mathrm{d}S_x\mathrm{d}t. \end{aligned} \tag{3.8}$$

Combining Equations (3.6),(3.7) and (3.8) we get

$$\begin{aligned} I_1 \geq\; & s\int\limits_Q |\nabla v(t,x)|^2\mathrm{d}x\mathrm{d}t - \frac{s^2}{2}\int\limits_Q |v(t,x)|^2\mathrm{d}x\mathrm{d}t + \frac{1+4\lambda^2}{32R^2}\int\limits_Q |v(t,x)|^2\mathrm{d}x\mathrm{d}t \\ &+ \frac{1}{2}\int\limits_\Omega |\nabla v(T,x)|^2\mathrm{d}x + \frac{\lambda}{2}\int\limits_\Sigma \omega\cdot\nu(x)|\partial_\nu v(t,x)|^2\mathrm{d}S_x\mathrm{d}t. \end{aligned} \tag{3.9}$$

Next we estimate $I_2$.

$$\begin{aligned}I_2 &\leq \int\limits_Q \Big|\Big( -2A(t,x)\cdot\nabla_x - 2\lambda\omega\cdot A(t,x) + \widetilde{q}(t,x)\Big)v(t,x)\Big|^2 \mathrm{d}x\mathrm{d}t \\ &\leq 2\|\widetilde{q}\|_\infty^2 \int_Q |v(t,x)|^2\mathrm{d}x\mathrm{d}t + 8\lambda^2\|A\|_\infty^2 \int_Q |v(t,x)|^2\mathrm{d}x\mathrm{d}t + 8\|A\|_\infty^2 \int\limits_Q |\nabla v(t,x)|^2\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{3.10}$$

Using (3.9) and (3.10) in (3.5) we get

$$\begin{aligned}I &\geq s\int\limits_Q |\nabla_x v(t,x)|^2\mathrm{d}x\mathrm{d}t - \frac{s^2}{2}\int\limits_Q |v(t,x)|^2\mathrm{d}x\mathrm{d}t + \frac{1+4\lambda^2}{32R^2}\int\limits_Q |v(t,x)|^2\mathrm{d}x\mathrm{d}t \\ &\quad + \frac{1}{2}\int\limits_\Omega |\nabla_x v(T,x)|^2\mathrm{d}x + \frac{\lambda}{2}\int\limits_\Sigma \omega\cdot\nu(x)|\partial_\nu v(t,x)|^2\mathrm{d}S_x\mathrm{d}t - 2\|\tilde{q}\|_\infty^2\int_Q |v(t,x)|^2\mathrm{d}x\mathrm{d}t \\ &\quad - 8\lambda^2\|A\|_\infty^2\int_Q |v(t,x)|^2\mathrm{d}x\mathrm{d}t - 8\|A\|_\infty^2\int\limits_Q |\nabla_x v(t,x)|^2\mathrm{d}x\mathrm{d}t. \\ &\geq \left(\frac{1+4\lambda^2}{32R^2} - \frac{s^2}{2} - 2\|\tilde{q}\|_\infty^2 - 8\lambda^2\|A\|_\infty^2\right)\int\limits_Q |v(t,x)|^2\mathrm{d}x\mathrm{d}t + \big(s - 8\|A\|_\infty^2\big)\int\limits_Q |\nabla_x v(t,x)|^2\mathrm{d}x\mathrm{d}t \\ &\quad + \frac{1}{2}\int\limits_\Omega |\nabla_x v(T,x)|^2\mathrm{d}x + \frac{\lambda}{2}\int\limits_\Sigma \omega\cdot\nu(x)|\partial_\nu v(t,x)|^2\mathrm{d}S_x\mathrm{d}t.\end{aligned}$$

Now since $\|A\|_\infty \leq \frac{1}{9R}$, therefore taking $\lambda$ large enough and using the Poincaré inequality, we have

$$\begin{aligned}\int\limits_Q |\mathcal{L}_\varphi v(t,x)|^2\mathrm{d}x\mathrm{d}t \geq C\Bigg(&\lambda^2\int\limits_Q |v(t,x)|^2\mathrm{d}x\mathrm{d}t + \int\limits_Q |\nabla_x v(t,x)|^2\mathrm{d}x\mathrm{d}t \\ &+ \frac{1}{2}\int\limits_\Omega |v(T,x)|^2\mathrm{d}x + \lambda\int\limits_\Sigma \omega\cdot\nu(x)|\partial_\nu v(t,x)|^2\mathrm{d}S_x\mathrm{d}t\Bigg)\end{aligned} \tag{3.11}$$

holds for large $\lambda$ and $C > 0$ depending only on $Q$, $A$ and $\widetilde{q}$. Now after substituting $v(t,x) = e^{-\varphi(t,x)}u(t,x)$ in (3.11), we get

$$\begin{aligned}&\lambda^2\int_Q e^{-2\varphi}|u(t,x)|^2\mathrm{d}x\mathrm{d}t + \int_Q e^{-2\varphi}|\nabla_x u(t,x)|^2\mathrm{d}x\mathrm{d}t + \int_\Omega e^{-2\varphi(T,x)}|u(T,x)|^2\mathrm{d}x \\ &\quad + \lambda\int_{\Sigma_{+,\omega}} e^{-2\varphi}|\partial_\nu u(t,x)|^2|\omega\cdot\nu(x)|\mathrm{d}S_x\mathrm{d}t \leq C\int\limits_Q e^{-2\varphi}|\mathcal{L}_{A,q}u(t,x)|^2\mathrm{d}x\mathrm{d}t \\ &\quad + C\lambda\int\limits_{\Sigma_{-,\omega}} e^{-2\varphi}|\partial_\nu u(t,x)|^2\ |\omega\cdot\nu(x)|\mathrm{d}S_x\mathrm{d}t.\end{aligned}$$

This completes the proof of Carleman estimate given by (3.1). □

## 4. Construction of geometric optics solutions

In this section, we construct the exponentially growing solution to

$$\mathcal{L}_{A,q}u(t,x)=0,\ (t,x)\in Q$$

and exponentially decaying solution to

$$\mathcal{L}^*_{A,q}u(t,x)=0,\ (t,x)\in Q$$

where $\mathcal{L}^*_{A,q}$ given by

$$\mathcal{L}^*_{A,q}:=-\partial_t-\sum_{j=1}^{n}\left(\partial_j-A_j(t,x)\right)^2+\widetilde{q}^*(t,x)$$

is a formal $L^2$ adjoint of the operator $\mathcal{L}_{A,q}$. We construct these solutions by using a Carleman estimate in a Sobolev space of negative order as used in [26] for elliptic case and in [39, 44] for hyperbolic case. Before going further following [39] we will give some definition and notation, which will be used later. For $m\in\mathbb{R}$, define space $L^2\left(0,T;H^m_\lambda(\mathbb{R}^n)\right)$ by

$$L^2\left(0,T;H^m_\lambda(\mathbb{R}^n)\right):=\Big\{u(t,\cdot)\in\mathcal{S}^{'}(\mathbb{R}^n):\left(\lambda^2+|\xi|^2\right)^{m/2}\widehat{u}(t,\xi)\in L^2(\mathbb{R}^n)\Big\}$$

with the norm

$$\|u\|^2_{L^2\left(0,T;H^m_\lambda(\mathbb{R}^n)\right)}:=\int\limits_0^T\int\limits_{\mathbb{R}^n}\left(\lambda^2+|\xi|^2\right)^m|\widehat{u}(t,\xi)|^2d\xi dt$$

where $\mathcal{S}^{'}(\mathbb{R}^n)$ denote the space of all tempered distribution on $\mathbb{R}^n$ and $\widehat{u}(t,\xi)$ is the Fourier transform with respect to space variable $x\in\mathbb{R}^n$. We define by

$$\langle D_x,\lambda\rangle^m u=\mathcal{F}_x^{-1}\left\{\left(\lambda^2+|\xi|^2\right)^{m/2}\mathcal{F}_x u\right\}$$

here $\mathcal{F}_x$ and $\mathcal{F}_x^{-1}$ denote the Fourier transform and inverse Fourier transform respectively with respect to space variable $x\in\mathbb{R}^n$. With this we define the symbol class $S^m_\lambda(\mathbb{R}^n)$ of order $m$ by

$$S^m_\lambda(\mathbb{R}^n):=\left\{C_\lambda\in C^\infty(\mathbb{R}^n\times\mathbb{R}^n):|\partial_x^\alpha\partial_\xi^\beta c_\lambda(x,\xi)|\leq C_{\alpha,\beta}\left(\lambda^2+|\xi|^2\right)^{m-|\beta|},\ \text{for multi-indices }\alpha,\beta\in\mathbb{N}^n\right\}.$$

With these notations and definitions, we state the main theorem of this section.

**Theorem 4.1.** *(1) (Exponentially growing solutions) Let $\mathcal{L}_{A,q}$ be as defined above. Then for $\lambda$ large there exists $v\in H^1\left(0,T;H^{-1}(\Omega)\right)\cap L^2\left(0,T;H^1(\Omega)\right)$ a solution to*

$$\begin{cases}\mathcal{L}_{A,q}v(t,x)=0,\ (t,x)\in Q,\\ v(0,x)=0,\ x\in\Omega\end{cases}$$

*of the following form*

$$v_g(t,x)=e^{\varphi}\Big(B_g(t,x)+R_g(t,x,\lambda)\Big)\tag{4.1}$$

*where for $\chi\in C_c^\infty((0,T))$ arbitrary, we have*

$$B_g(t,x)=\chi(t)e^{-\mathrm{i}(t\tau+x\cdot\xi)}\exp\left(\int_0^\infty\omega\cdot A(t,x+s\omega)\mathrm{d}s\right)\tag{4.2}$$

*and $R_g(t,x,\lambda)$ satisfies the following*

$$R_g(0,x,\lambda)=0,\ \text{for }x\in\Omega\text{ and }\|R_g\|_{L^2\left(0,T;H^1_\lambda(\mathbb{R}^n)\right)}\leq C.\tag{4.3}$$

(2) *(Exponentially decaying solutions) Let $\mathcal{L}^*_{A,q}$ be as before. Then for $\lambda$ large there exists $v \in H^1(0,T;H^{-1}(\Omega)) \cap L^2(0,T;H^1(\Omega))$ a solution to*

$$\begin{cases} \mathcal{L}^*_{A,q} v(t,x) = 0, \ (t,x) \in Q, \\ v(T,x) = 0, \ x \in \Omega \end{cases}$$

*of the following form*

$$v_d(t,x) = e^{-\varphi}\Big(B_d(t,x) + R_d(t,x,\lambda)\Big) \tag{4.4}$$

*where for $\chi \in C_c^\infty((0,T))$ arbitrary, we have*

$$B_d(t,x) = \chi(t)\exp\left(-\int_0^\infty \omega \cdot A(t,x+s\omega)\mathrm{d}s\right) \tag{4.5}$$

*and $R_d(t,x,\lambda)$ satisfies the following*

$$R_d(T,x,\lambda) = 0, \ \text{for } x \in \Omega \text{ and } \|R_d\|_{L^2\left(0,T;H^1_\lambda(\mathbb{R}^n)\right)} \leq C. \tag{4.6}$$

Proof of the above theorem is based on a Carleman estimate in a Sobolev space of negative order. To prove the Carleman estimate stated in Proposition 4.2, we follow the arguments similar to one used in [26, 39, 44] for elliptic and hyperbolic inverse problems.

**Proposition 4.2.** *Let $\varphi, A$ and $q$ be as in Theorem 3.1. Then for $\lambda$ large enough, we have*

(1) *(Interior Carleman estimate for $\mathcal{L}^*_\varphi$ ) Let $\mathcal{L}^*_\varphi := e^{\varphi}\mathcal{L}^*_{A,q}e^{-\varphi}$, then there exists a constant $C > 0$ independent of $\lambda$ and $v$ such that*

$$\|v\|_{L^2(0,T;L^2(\mathbb{R}^n))} \leq C\|\mathcal{L}^*_\varphi v\|_{L^2\left(0,T;H^{-1}_\lambda(\mathbb{R}^n)\right)}, \tag{4.7}$$

*holds for $v \in C^1([0,T];C_c^\infty(\Omega))$ satisfying $v(T,x) = 0$ for $x \in \Omega$.*

(2) *(Interior Carleman estimate for $\mathcal{L}_\varphi$ ) Let $\mathcal{L}_\varphi$ be as before then there exists a constant $C > 0$ independent of $\lambda$ and $v$ such that*

$$\|v\|_{L^2(0,T;L^2(\mathbb{R}^n))} \leq C\|\mathcal{L}_\varphi v\|_{L^2\left(0,T;H^{-1}_\lambda(\mathbb{R}^n)\right)} \tag{4.8}$$

*holds for $v \in C^1([0,T];C_c^\infty(\Omega))$ satisfying $v(0,x) = 0$ for $x \in \Omega$.*

*Proof.* (1) (Proof for (4.7)) Since

$$\mathcal{L}^*_\varphi = e^{\varphi}\mathcal{L}^*_{A,q}e^{-\varphi}$$

therefore we have

$$\begin{aligned} \mathcal{L}^*_\varphi v &= e^{\varphi}\Big(-\partial_t - \Delta + 2A(t,x)\cdot\nabla_x + \widetilde{q}^*(t,x)\Big)e^{-\varphi}v(t,x) \\ &= \Big[-\partial_t - \Delta + 2A(t,x)\cdot\nabla_x + \widetilde{q}^*(t,x) - 2\lambda\omega\cdot A(t,x) + 2\lambda\omega\cdot\nabla_x\Big]v(t,x) \end{aligned}$$

where

$$\widetilde{q}^*(t,x) := \overline{q}(t,x) + \nabla_x\cdot A(t,x) - |A(t,x)|^2.$$

Writing $\mathcal{L}^*_\varphi$ as

$$\mathcal{L}^*_\varphi v := P^*_1 v + P^*_2 v + P^*_3 v$$

where

$$P^*_1 := -\Delta, \ P^*_2 := -\partial_t + 2\lambda\omega\cdot\nabla_x \text{ and } P^*_3 := 2A(t,x)\cdot\nabla_x - 2\lambda\omega\cdot A(t,x) + \widetilde{q}^*(t,x). \tag{4.9}$$

Now from (3.4), we have $P_1^* = P_1$ and $P_2^* = -P_2$. Hence using the arguments similar to Theorem 3.1, we have

$$\int_Q |\nabla v(t,x)|^2 \mathrm{d}x\mathrm{d}t + \lambda^2 \int_Q |v(t,x)|^2 \mathrm{d}x\mathrm{d}t \leq C \int_Q |\mathcal{L}_\varphi^* v(t,x)|^2 \mathrm{d}x\mathrm{d}t$$

for some constant $C > 0$ independent of $\lambda$ and $v$. The above estimate can be written in compact form as

$$\|v\|_{L^2\left(0,T;H^1_\lambda(\mathbb{R}^n)\right)} \leq C\|\mathcal{L}_\varphi^* v\|_{L^2(Q)}, \text{ for some constant } C \text{ independent of } \lambda \text{ and } v. \tag{4.10}$$

Next using the pseudodifferential operators techniques, we shift the index by $-1$ in the above estimate. Let us denote by $\widetilde{\Omega}$ a bounded open subset of $\mathbb{R}^n$ such that $\overline{\Omega} \subset \widetilde{\Omega}$. Fix $w \in C^1\left([0,T]; C_c^\infty(\Omega)\right)$ satisfying $w(T,x) = 0$ and consider the following

$$\langle D_x, \lambda\rangle^{-1}(P_1^* + P_2^*)\langle D_x, \lambda\rangle w.$$

Using the composition of pseudodifferential operators [31, Theorem 18.1.8] we have

$$\langle D_x, \lambda\rangle^{-1}(P_1^* + P_2^*)\langle D_x, \lambda\rangle w = (P_1^* + P_2^*)w. \tag{4.11}$$

Using (4.11) and (3.9) we have

$$\begin{aligned} &\|(P_1^* + P_2^*)\langle D_x, \lambda\rangle w\|_{L^2\left(0,T;H^{-1}_\lambda(\mathbb{R}^n)\right)} = \|\langle D_x, \lambda\rangle^{-1}(P_1^* + P_2^*)\langle D_x, \lambda\rangle w\|_{L^2(0,T;L^2(\mathbb{R}^n))} \\ &= \|(P_1^* + P_2^*)w\|_{L^2(0,T;L^2(\mathbb{R}^n))} \geq \sqrt{s}\|\nabla w\|_{L^2(0,T;L^2(\mathbb{R}^n))} + \lambda\|w\|_{L^2(0,T;L^2(\mathbb{R}^n))} \end{aligned} \tag{4.12}$$

holds for $\lambda$ large. Now consider

$$\begin{aligned} \|P_3^*\langle D_x, \lambda\rangle w\|_{L^2\left(0,T;H^{-1}_\lambda(\mathbb{R}^n)\right)} &\leq 2\bigg( \|(\lambda\omega \cdot A(t,x)\langle D_x, \lambda\rangle w\|_{L^2\left(0,T;H^{-1}_\lambda(\mathbb{R}^n)\right)} \\ &+ \|A(t,x) \cdot \nabla_x \langle D_x, \lambda\rangle w\|_{L^2\left(0,T;H^{-1}_\lambda(\mathbb{R}^n)\right)} + \|\widetilde{q}^* w\|_{L^2\left(0,T;H^{-1}_\lambda(\mathbb{R}^n)\right)} \bigg). \end{aligned}$$

Using the boundedness of the coefficients, we have

$$\begin{aligned} \|P_3^*\langle D_x, \lambda\rangle w\|_{L^2\left(0,T;H^{-1}_\lambda(\mathbb{R}^n)\right)} &\leq 2\bigg( \lambda\|A\|_\infty \|w\|_{L^2(0,T;L^2(\mathbb{R}^n))} \\ &+ \|A\|_\infty \|\nabla w\|_{L^2(0,T;L^2(\mathbb{R}^n))} + \|\tilde{q}\|_\infty \|w\|_{L^2(0,T;L^2(\mathbb{R}^n))} \bigg). \end{aligned}$$

Hence using the inequality as used in (3.5) we get

$$\|\mathcal{L}_\varphi^*\langle D, \lambda\rangle w\|_{L^2\left(0,T;H^{-1}_\lambda(\mathbb{R}^n)\right)} \geq C\|w\|_{\text{Ł}^2\left(0,T;H^1_\lambda(\mathbb{R}^n)\right)}.$$

Now let $\chi \in C_c^\infty(\widetilde{\Omega})$ such that $\chi = 1$ in $\overline{\Omega}_1$ where $\overline{\Omega} \subset \Omega_1 \subset \widetilde{\Omega}$. Fix $w = \chi\langle D, \lambda\rangle^{-1} v$ in the above equation and using

$$\|(1-\chi)\langle D, \lambda\rangle^{-1} v\|_{L^2(0,T;H^m_\lambda(\mathbb{R}^n))} \leq \frac{C}{\lambda^2}\|v\|_{L^2(0,T;L^2(\mathbb{R}^n))}$$

and

$$\begin{aligned}\|v\|_{L^2(0,T;L^2(\mathbb{R}^n))} &= \|\langle D,\lambda\rangle^{-1}v\|_{L^2(0,T;H^1_\lambda(\mathbb{R}^n))}\\ &\le \|w\|_{L^2(0,T;H^1_\lambda(\mathbb{R}^n))} + \|(1-\chi)\langle D,\lambda\rangle^{-1}v\|_{L^2(0,T;H^1_\lambda(\mathbb{R}^n))}\\ &\le \|w\|_{L^2(0,T;H^1_\lambda(\mathbb{R}^n))} + \frac{C}{\lambda^2}\|v\|_{L^2(0,T;L^2(\mathbb{R}^n))}\end{aligned}$$

we get

$$\begin{aligned}\|\mathcal{L}^*_\varphi v\|_{L^2\left(0,T;H^{-1}_\lambda(\mathbb{R}^n)\right)} &\ge \|\mathcal{L}^*_\varphi\langle D,\lambda\rangle w\|_{L^2\left(0,T;H^{-1}_\lambda(\mathbb{R}^n)\right)} - \frac{C}{\lambda^2}\|v\|_{L^2(0,T;L^2(\mathbb{R}^n))}\\ &\ge \|w\|_{L^2(0,T;H^1_\lambda(\mathbb{R}^n))} - \frac{C}{\lambda^2}\|v\|_{L^2(0,T;L^2(\mathbb{R}^n))} \ge C\|v\|_{L^2(0,T;L^2(\mathbb{R}^n))}\end{aligned}$$

for large $\lambda$. Thus finally, we have

$$\|v\|_{L^2(0,T;L^2(\mathbb{R}^n))} \le C\|\mathcal{L}^*_\varphi v\|_{L^2\left(0,T;H^{-1}_\lambda(\mathbb{R}^n)\right)}$$

holds for $v \in C^1([0,T];C_c^\infty(\Omega))$ such that $v(T,x)=0$ and $\lambda$ large.

(2) (Proof for (4.8)) follows by exactly the same argument as that for (4.7).

□

**Proposition 4.3.** *Let $\varphi$, $A$ and $q$ be as in Theorem 3.1.*

*(1) (Existence of solution to $\mathcal{L}_{A,q}$ ) For $\lambda>0$ large enough and $v\in L^2(Q)$ there exists a solution $u\in H^1(0,T;H^{-1}(\Omega))\cap L^2(0,T;H^1(\Omega))$ of*

$$\begin{cases} \mathcal{L}_\varphi u(t,x) = v(t,x),\ (t,x)\in Q,\\ u(0,x)=0;\ x\in\Omega\end{cases}$$

*and it satisfies*

$$\|u\|_{L^2(0,T;H^1(\Omega))} \le C\|v\|_{L^2(Q)} \tag{4.13}$$

*where $C>0$ is a constant independent of $\lambda$.*

*(2) (Existence of solution to $\mathcal{L}^*_{A,q}$) For $\lambda>0$ large enough and $v\in L^2(Q)$ there exists a solution $u\in H^1(0,T;H^{-1}(\Omega))\cap L^2(0,T;H^1(\Omega))$ of*

$$\begin{cases} \mathcal{L}^*_\varphi u(t,x) = v(t,x),\ (t,x)\in Q,\\ u(T,x)=0,\ x\in\Omega\end{cases}$$

*and it satisfies*

$$\|u\|_{L^2(0,T;H^1(\Omega))} \le C\|v\|_{L^2(Q)} \tag{4.14}$$

*where $C>0$ is a constant independent of $\lambda$.*

*Proof.* We will give the proof for existence of solution to $\mathcal{L}_{A,q}$ and the proof for $\mathcal{L}^*_{A,q}$ follows by using similar arguments. The proof is based on the standard functional analysis arguments. Consider the space $S := \{\mathcal{L}^*_\varphi u : u\in C^1([0,T];C_c^\infty(\Omega))$ and $u(T,x)=0\}$ as a subspace of $L^2\left(0,T;H^{-1}_\lambda(\mathbb{R}^n)\right)$. Define the linear operator $T$ on $S$ by

$$T(\mathcal{L}^*_\varphi z) = \int_Q z(t,x)v(t,x)\mathrm{d}t\mathrm{d}x,\quad \text{for } v\in L^2(Q).$$

Now using the Carleman estimates (4.7), we have

$$|T(\mathcal{L}_\varphi^* z)| \leq \|z\|_{L^2(Q)}\|v\|_{L^2(Q)} \leq C\|v\|_{L^2(Q)}\|\mathcal{L}_\varphi^* z\|_{L^2\left(0,T;H_\lambda^{-1}(\mathbb{R}^n)\right)}$$

holds for $z \in C^1\left([0,T];C_c^\infty(\Omega)\right)$ with $z(T,x)=0$. Hence using the Hahn-Banach theorem, we can extend the linear operator $T$ to $L^2\left(0,T;H_\lambda^{-1}(\mathbb{R}^n)\right)$. We denote the extended map as $T$ and it satisfies

$$\|T\| \leq C\|v\|_{L^2(Q)}.$$

Since $T$ is bounded linear functional on $L^2\left(0,T;H_\lambda^{-1}(\mathbb{R}^n)\right)$ therefore using the Riesz representation theorem there exists a unique $u \in L^2(0,T;H_\lambda^1(\mathbb{R}^n))$ such that

$$T(f) = \langle f, u\rangle_{L^2\left(0,T;H_\lambda^{-1}(\mathbb{R}^n)\right),L^2\left(0,T;H_\lambda^1(\mathbb{R}^n)\right)} \text{ for } f \in L^2\left(0,T;H_\lambda^{-1}(\mathbb{R}^n)\right) \tag{4.15}$$

with $\|u\|_{L^2(0,T;H_\lambda^1(\mathbb{R}^n))} \leq C\|v\|_{L^2(Q)}$. Now for $z \in C^1\left([0,T];C_c^\infty(\Omega)\right)$ satisfying $z(T,x)=0$. Choosing $f = \mathcal{L}_\varphi^* z$ in the above equation, we get $\mathcal{L}_\varphi u = v$. Using the expression for $\mathcal{L}_\varphi$ from (3.3) and the fact that $u \in L^2\left(0,T;H^1(\Omega)\right)$ and $v \in L^2(Q)$, we get that $\partial_t u \in L^2\left(0,T;H^{-1}(\Omega)\right)$. Hence we have $u \in H^1\left(0,T;H^{-1}(\Omega)\right) \cap L^2\left(0,T;H^1(\Omega)\right)$.

Next we will show that $u(0,x)=0$ for $x \in \Omega$. To prove this we choose $f = \mathcal{L}_\varphi^* z$ where $z \in C^1\left([0,T];C_c^\infty(\Omega)\right)$ and $z(T,x)=0$. Using this choice of $f$ in (4.15), we have

$$\int_Q \mathcal{L}_\varphi^* z(t,x)u(t,x)\mathrm{d}x\mathrm{d}t = \int_Q z(t,x)v(t,x)\mathrm{d}x\mathrm{d}t.$$

Now using integration by parts and the fact that $\mathcal{L}_\varphi u = v$, we get

$$\int_\Omega u(0,x)z(0,x)\mathrm{d}x = 0.$$

The above identity holds for any $z \in C^1\left([0,T];C_c^\infty(\Omega)\right)$ satisfying $z(T,x)=0$. Therefore, we conclude that $u(0,x)=0$ for $x \in \Omega$. This completes the proof of first part of Proposition 4.3 .

□

4.1. **Proof of the Theorem 4.1.** Using expressions $v_g$ and $B_g$ from (4.1) and (4.2) respectively and

$$\mathcal{L}_{A,q}v_g(t,x) = 0,$$

we have the equation for $R_g$ is

$$\mathcal{L}_\varphi R_g(t,x,\lambda) = -\mathcal{L}_{A,q}B_g(t,x),$$

where $\mathcal{L}_{A,q}B_g(t,x) \in L^2(Q)$. Next using Proposition 4.3, there exists $R_g \in L^2\left(0,T;H^1(\Omega)\right) \cap H^1\left(0,T;H^{-1}(\Omega)\right)$ solution to

$$\begin{cases} \mathcal{L}_\varphi R_g(t,x,\lambda) = -\mathcal{L}_{A,q}B_g(t,x),\ (t,x) \in Q, \\ R_g(0,x,\lambda) = 0,\ x \in \Omega \end{cases}$$

and it satisfies the following estimate

$$\|R_g\|_{L^2(0,T;H^1(\Omega))} \leq C$$

where $C$ is a constant independent of $\lambda$. This completes the construction of solution for $\mathcal{L}_{A,q}u = 0$ and existence of the solution for $\mathcal{L}_{A,q}^* v = 0$, follows in a similar way.

## 5. Integral identity

This section is devoted to proving an integral identity which will be used to prove the main result of this article. We derive this identity by using the geometric optics solutions constructed in §4. Let $u_i$ be the solutions to the following initial boundary value problems with vector field coefficient $A^{(i)}$ and scalar potential $q_i$ for $i = 1, 2$.

$$\begin{cases} \mathcal{L}_{A^{(i)},q_i} u_i(t,x) = 0, \ (t,x) \in Q \\ u_i(0,x) = 0, \ x \in \Omega \\ u_i(t,x) = f(t,x), \ (t,x) \in \Sigma. \end{cases} \tag{5.1}$$

Let us denote

$$\begin{aligned} u(t,x) &:= (u_1 - u_2)(t,x) \\ A(t,x) &:= \left(A^{(1)} - A^{(2)}\right)(t,x) := (A_1(t,x), \cdots, A_n(t,x)) \\ \widetilde{q}_i(t,x) &:= -\nabla_x \cdot A^{(i)}(t,x) - |A^{(i)}(t,x)|^2 + q_i(t,x) \\ \widetilde{q}(t,x) &:= \widetilde{q}_1(t,x) - \widetilde{q}_2(t,x). \end{aligned} \tag{5.2}$$

Then $u$ is solution to the following initial boundary value problem:

$$\begin{cases} \mathcal{L}_{A^{(1)},q_1} u(t,x) = 2A(t,x) \cdot \nabla_x u_2(t,x) + \widetilde{q} u_2(t,x), \ (t,x) \in Q \\ u(0,x) = 0, \ x \in \Omega \\ u(t,x) = 0, \ (t,x) \in \Sigma. \end{cases} \tag{5.3}$$

Let $v(t,x)$ of the form given by (4.4) be the solution to following equation

$$\mathcal{L}^*_{A^{(1)},q_1} v(t,x) = 0, \ (t,x) \in Q. \tag{5.4}$$

Also let $u_2$ of the form given by (4.1) be solution to the following equation

$$\mathcal{L}_{A^{(2)},q_2} u_2(t,x) = 0, \ (t,x) \in Q. \tag{5.5}$$

Since the right hand side of (5.3) lies in $L^2(Q)$ therefore using ([23], Theorem 1.43) we have $u \in L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))$ and $\partial_\nu u \in L^2\left(0,T;H^{1/2}(\Sigma)\right)$. Next consider the following

$$\begin{aligned} \Big\langle \left(\Lambda_{A^{(1)},q_1} - \Lambda_{A^{(2)},q_2}\right)(f), v|_\Sigma \Big\rangle_{\mathcal{H}^*_T,\mathcal{H}_T} &= \Big\langle \mathcal{N}_{A^{(1)},q_1} u_1 - \mathcal{N}_{A^{(2)},q_2} u_2, v|_\Sigma \Big\rangle_{\mathcal{H}^*_T,\mathcal{H}_T} \\ &= \int_Q \left(-u_1 \partial_t \overline{v} + \nabla_x u_1 \cdot \nabla_x \overline{v} + 2u_1 A^{(1)} \cdot \nabla_x \overline{v} + (\nabla_x \cdot A^{(1)}) u_1 \overline{v} - |A^{(1)}|^2 u_1 v + q_1 u_1 \overline{v}\right) \mathrm{d}x\mathrm{d}t \\ &\quad - \int_Q \left(-u_2 \partial_t \overline{v} + \nabla_x u_2 \cdot \nabla_x \overline{v} + 2u_2 A^{(2)} \cdot \nabla_x \overline{v} + (\nabla_x \cdot A^{(2)}) u_2 \overline{v} - |A^{(2)}|^2 u_2 v + q_2 u_2 \overline{v}\right) \mathrm{d}x\mathrm{d}t. \end{aligned}$$

After following the arguments used in [[14], see Proposition 2.3], we get that

$$\Big\langle \left(\Lambda_{A^{(1)},q_1} - \Lambda_{A^{(2)},q_2}\right)(f), v|_\Sigma \Big\rangle_{\mathcal{H}^*_T,\mathcal{H}_T} = \int_Q \left(2A(t,x) \cdot \nabla_x u_2(t,x) + \widetilde{q}(t,x) u_2(t,x)\right) \overline{v(t,x)} \mathrm{d}x\mathrm{d}t. \tag{5.6}$$

Also multiplying (5.3) by $\overline{v}(t,x)$ and integrating over $Q$, we have

$$\int_Q \left(2A(t,x)\cdot\nabla_x u_2(t,x)+\widetilde{q}(t,x)u_2(t,x)\right)\overline{v(t,x)}\mathrm{d}x\mathrm{d}t=\int_Q \mathcal{L}_{A^{(1)},q_1}u(t,x)\overline{v(t,x)}\mathrm{d}x\mathrm{d}t$$

$$=\int_Q u(t,x)\overline{\mathcal{L}^*_{A^{(1)},q_1}v(t,x)}\mathrm{d}x\mathrm{d}t-\int_\Sigma \partial_\nu u(t,x)\overline{v(t,x)}\mathrm{d}S_x\mathrm{d}t+\int_\Omega u(T,x)\overline{v(T,x)}\mathrm{d}x$$

where in deriving the above identity we have used the following: $u|_\Sigma=0$, $u|_{t=0}=0$ and $A^{(1)}=A^{(2)}$ on $\Sigma$. Now using Equation (5.6) and the fact that $\mathcal{L}^*_{A^{(1)},q_1}v(t,x)=0$ in $Q$, with $v(T,x)=0$ in $\Omega$, we get,

$$\left\langle \left(\Lambda_{A^{(1)},q_1}-\Lambda_{A^{(2)},q_2}\right)(f),v|_\Sigma\right\rangle_{\mathcal{H}^*_T,\mathcal{H}_T}=-\int_\Sigma \partial_\nu u(t,x)\overline{v(t,x)}\mathrm{d}S_x\mathrm{d}t. \tag{5.7}$$

This gives us

$$\left(\Lambda_{A^{(1)},q_1}-\Lambda_{A^{(2)},q_2}\right)(f)|_\Sigma=-\partial_\nu u|_\Sigma. \tag{5.8}$$

Using (2.2), we have $\partial_\nu u|_G=0$. Finally using Equations (5.7), (5.8) and $\partial_\nu u|_G=0$, in (5.6), we get

$$\int_Q \left(2A(t,x)\cdot\nabla_x u_2(t,x)+\widetilde{q}(t,x)u_2(t,x)\right)\overline{v(t,x)}\mathrm{d}x\mathrm{d}t=-\int_{\Sigma\setminus G}\partial_\nu u(t,x)\overline{v(t,x)}\mathrm{d}S_x\mathrm{d}t. \tag{5.9}$$

Next we need to estimate the right hand side of above equation. This we will do in the following lemma:

**Lemma 5.1.** *Let $u_i$ for $i=1,2$ solutions to* (5.1) *with $u_2$ of the form* (4.1)*. Let $u=u_1-u_2$ and $v$ be of the form* (4.4)*. Then*

$$\Big|\int_{\Sigma\setminus G}\partial_\nu u(t,x)\overline{v(t,x)}\mathrm{d}S_x\mathrm{d}t\Big|\le C\lambda^{1/2} \tag{5.10}$$

*for all $\omega\in\mathbb{S}^{n-1}$ such that $|\omega-\omega_0|\le\epsilon$.*

*Proof.* Using the expression of $v$ from (4.4), in the right-hand side of (5.9), we have

$$\begin{aligned}\Big|\int_{\Sigma\setminus G}\partial_\nu u(t,x)\overline{v(t,x)}\mathrm{d}S_x\mathrm{d}t\Big|^2&\le\Big|\int_{\Sigma\setminus G}\partial_\nu u(t,x)e^{-\varphi(t,x)}\left(\overline{B_d}(t,x)+\overline{R_d}(t,x)\right)\mathrm{d}S_x\mathrm{d}t\Big|^2\\&\le C\left(1+\|R_d\|^2_{L^2(\Sigma)}\right)\int_{\Sigma\setminus G}e^{-2\varphi(t,x)}|\partial_\nu u(t,x)|^2\mathrm{d}S_x\mathrm{d}t\\&\le C\left(1+\|R_d\|_{L^2(0,T;H^1(\Omega))}\right)\int_{\Sigma\setminus G}e^{-2\varphi(t,x)}|\partial_\nu u(t,x)|^2\mathrm{d}S_x\mathrm{d}t\end{aligned}$$

where in the last step of above inequality we have used the trace theorem. Now using Equation (4.6), we get

$$\Big|\int_{\Sigma\setminus G}\partial_\nu u(t,x)\overline{v(t,x)}\mathrm{d}S_x\mathrm{d}t\Big|^2\le C\int_{\Sigma\setminus G}e^{-2\varphi(t,x)}|\partial_\nu u(t,x)|^2\mathrm{d}S_x\mathrm{d}t.$$

For $\varepsilon > 0$, define

$$\partial\Omega_{+,\varepsilon,\omega} := \{x \in \partial\Omega : \nu(x)\cdot\omega > \varepsilon\} \text{ and } \Sigma_{+,\varepsilon,\omega} := (0,T)\times\partial\Omega_{+,\varepsilon,\omega}$$

then from the definition of $G$ it follows that $\Sigma\setminus G \subseteq \Sigma_{+,\varepsilon,\omega}$ for all $\omega$ with $|\omega-\omega_0|\leq\varepsilon$. Using this we obtain

$$\begin{aligned}\int\limits_{\Sigma\setminus G} e^{-2\varphi(t,x)}|\partial_\nu u(t,x)|^2 \mathrm{d}S_x\mathrm{d}t &\leq \int\limits_{\Sigma_{+,\varepsilon,\omega}} e^{-2\varphi(t,x)}|\partial_\nu u(t,x)|^2\mathrm{d}S_x\mathrm{d}t\\ &= \frac{1}{\lambda\varepsilon}\int\limits_{\Sigma_{+,\varepsilon,\omega}} \lambda\varepsilon e^{-2\varphi(t,x)}|\partial_\nu u(t,x)|^2\mathrm{d}S_x\mathrm{d}t, \text{ for } \omega\in\mathbb{S}^{n-1} \text{ near } \omega_0\in\mathbb{S}^{n-1}.\end{aligned}$$

Now $\lambda\epsilon \leq \partial_\nu\varphi(t,x)$ for $(t,x)\in\Sigma_{+,\epsilon,\omega}$ and $\omega\in\mathbb{S}^{n-1}$ with $|\omega-\omega_0|\leq\epsilon$. Using this in above equation, we get

$$\int\limits_{\Sigma\setminus G} e^{-2\varphi(t,x)}|\partial_\nu u(t,x)|^2\mathrm{d}S_x\mathrm{d}t \leq \frac{1}{\lambda\varepsilon}\int\limits_{\Sigma_{+,\varepsilon,\omega}} \partial_\nu\varphi e^{-2\varphi(t,x)}|\partial_\nu u(t,x)|^2\mathrm{d}S_x\mathrm{d}t,$$

for $\omega\in\mathbb{S}^{n-1}$near $\omega_0\in\mathbb{S}^{n-1}$. Now using the Carleman estimate (3.1) and Equation (5.3), we get

$$\Big|\int\limits_{\Sigma\setminus G}\partial_\nu u(t,x)\overline{v(t,x)}\mathrm{d}S_x\mathrm{d}t\Big|^2 \leq C\lambda^{-1}\int\limits_{Q} e^{-2\varphi(t,x)}|(2A(t,x)\cdot\nabla u_2(t,x)+\widetilde{q}(t,x)u_2(t,x))|^2\mathrm{d}x\mathrm{d}t. \tag{5.11}$$

Using expression for $u_2$ from (4.1) and Equation (4.3), we have

$$\int\limits_{Q} e^{-2\varphi(t,x)}|(2A(t,x)\cdot\nabla u_2(t,x)+\widetilde{q}(t,x)u_2(t,x))|^2\mathrm{d}x\mathrm{d}t \leq C\lambda^2.$$

Hence using this in (5.11), we get

$$\Big|\int\limits_{\Sigma\setminus G}\partial_\nu u(t,x)\overline{v(t,x)}\mathrm{d}S_x\mathrm{d}t\Big| \leq C\lambda^{1/2}, \text{ for } \omega\in\mathbb{S}^{n-1} \text{ such that } |\omega-\omega_0|\leq\epsilon.$$

This completes the proof of lemma. ☐

## 6. Proof of theorem 2.1 and Corollary 2.2

In this section, we prove the uniqueness results. Since from (5.9), we have

$$\int\limits_{Q}(2A(t,x)\cdot\nabla_x u_2(t,x)+\widetilde{q}(t,x)u_2(t,x))\,\overline{v(t,x)}\mathrm{d}x\mathrm{d}t = -\int\limits_{\Sigma\setminus G}\partial_\nu u(t,x)\overline{v(t,x)}\mathrm{d}S_x\mathrm{d}t.$$

Now using Equation (5.10), we have

$$\Big|\int\limits_{Q}(2A(t,x)\cdot\nabla_x u_2(t,x)+\widetilde{q}(t,x)u_2(t,x))\,\overline{v(t,x)}\mathrm{d}x\mathrm{d}t\Big| \leq C\lambda^{1/2}.$$

After dividing the above equation by $\lambda$ and taking $\lambda \to \infty$, we have

$$\lim_{\lambda\to\infty}\left(\frac{1}{\lambda}\int\limits_{Q}\left(2A(t,x)\cdot\nabla_x u_2(t,x)+\widetilde{q}(t,x)u_2(t,x)\right)\overline{v(t,x)}\mathrm{d}x\mathrm{d}t\right)=0. \tag{6.1}$$

Next using the expression for $u_2$ and $v$ from (4.1) and (4.4) respectively, we have

$$\int\limits_{Q}\omega\cdot A(t,x)B_g(t,x)\overline{B_d(t,x)}\mathrm{d}x\mathrm{d}t=0,\ \text{for all}\ \ \omega\in\mathbb{S}^{n-1}\ \text{such that}\ |\omega-\omega_0|\leq\epsilon.$$

This after using the expressions for $B_g(t,x)$ and $B_d(t,x)$ from Equations (4.2) and (4.5) respectively, we get

$$\int\limits_{Q}\omega\cdot A(t,x)\chi^2(t)e^{-i\xi\cdot x-i\tau t}\exp\left(-\int\limits_{0}^{\infty}\omega\cdot A(t,x+s\omega)\mathrm{d}s\right)\mathrm{d}x\mathrm{d}t=0, \tag{6.2}$$

for $\omega\in\mathbb{S}^{n-1}$ with $|\omega-\omega_0|\leq\epsilon$. Since the above identity holds for all $\chi\in C_c^{\infty}(0,T)$, therefore we get

$$\int\limits_{\mathbb{R}^n}\omega\cdot A(t,x)(x)e^{-i\xi\cdot x}\exp\left(-\int\limits_{0}^{\infty}\omega\cdot A(t,x+s\omega)ds\right)\mathrm{d}x=0 \tag{6.3}$$

where $\xi\cdot\omega=0$ for all $\omega$ with $|\omega-\omega_0|\leq\epsilon$. Now decompose $\mathbb{R}^n=\mathbb{R}\omega\oplus\omega^{\perp}$ and using this in the above equation, we have

$$\int\limits_{\omega^{\perp}}e^{-i\xi\cdot k}\left(\int\limits_{\mathbb{R}}\omega\cdot A(t,k+\tau\omega)\exp\left(-\int\limits_{0}^{\infty}\omega\cdot A\left(t,k+\tau\omega+s\omega\right)\mathrm{d}s\right)\mathrm{d}\tau\right)\mathrm{d}k=0,\ \text{for}\ \omega\ \text{with}\ |\omega-\omega_0|\leq\epsilon$$

here $\mathrm{d}k$ denotes the Lebesgue measure on $\omega^{\perp}$. After substituting $\tau+s=\tilde{s}$, we get

$$\int\limits_{\omega^{\perp}}e^{-i\xi\cdot k}\left(\int\limits_{\mathbb{R}}\omega\cdot A(t,k+\tau\omega)\exp\left(-\int\limits_{\tau}^{\infty}\omega\cdot A(t,k+\tilde{s}\omega)\mathrm{d}\tilde{s}\right)\mathrm{d}\tau\right)\mathrm{d}k=0,\ \text{for}\ \omega\ \text{with}\ |\omega-\omega_0|\leq\epsilon. \tag{6.4}$$

Now

$$\begin{aligned}
&\int\limits_{\omega^{\perp}}e^{-i\xi\cdot k}\left(\int\limits_{\mathbb{R}}\omega\cdot A(t,k+\tau\omega)\exp\left(-\int\limits_{\tau}^{\infty}\omega\cdot A(t,k+\tilde{s}\omega)\mathrm{d}\tilde{s}\right)\mathrm{d}\tau\right)\mathrm{d}k\\
&=\int\limits_{\omega^{\perp}}e^{-i\xi\cdot k}\int\limits_{\mathbb{R}}\frac{\partial}{\partial\tau}\exp\left(-\int\limits_{\tau}^{\infty}\omega\cdot A(t,k+s\omega)\mathrm{d}s\right)\mathrm{d}\tau\mathrm{d}k\\
&=\int\limits_{\omega^{\perp}}e^{-i\xi\cdot k}\left(1-\exp\left(-\int\limits_{\mathbb{R}}\omega\cdot A(t,k+s\omega)\mathrm{d}s\right)\right)\mathrm{d}k.
\end{aligned}$$

Combining this with (6.4), we get

$$\int_{\mathbb{R}} \omega \cdot A(t, k + s\omega)\mathrm{d}s = 0, \text{ for } k \in \omega^{\perp} \text{ with } |\omega - \omega_0| \leq \epsilon.$$

Now using the decomposition $\mathbb{R}^n = \mathbb{R}\omega \oplus \omega^{\perp}$ in the above equation, we get

$$\int_{\mathbb{R}} \omega \cdot A(t, x + s\omega)\mathrm{d}s = 0, \text{ for } x \in \mathbb{R}^n \text{ with } |\omega - \omega_0| \leq \epsilon. \tag{6.5}$$

Thus we have the ray transform of vector field $A$ is vanishing in a very small enough neighbourhood of fixed direction $\omega_0$. In order to get the uniqueness for vector field term $A$, we need to invert this ray transform which we will do in the following lemma:

**Lemma 6.1.** *Let $n \geq 2$ and $F = (F_1, F_2, \cdots, F_n)$ be a real-valued time-dependent vector field with $F_j \in C_c^{\infty}(Q)$ for all $1 \leq j \leq n$. Suppose for each $t \in (0, T)$ we have*

$$IF(t, x, \omega) := \int_{\mathbb{R}} \omega \cdot F(t, x + s\omega)\mathrm{d}s = 0$$

*for all $\omega \in \mathbb{S}^{n-1}$ with $|\omega - \omega_0| \leq \epsilon$, for some $\epsilon > 0$ and for all $x \in \mathbb{R}^n$. Then for each $t \in (0, T)$ there exists a $\Phi(t, \cdot) \in C_c^{\infty}(\Omega)$ such that $F(t, x) = \nabla_x \Phi(t, x)$.*

*Proof.* The proof uses the arguments similar to the one used in [44, 48, 52] for the case of light ray transforms. We assume that $t \in (0, T)$ is arbitrary but fixed. We have the ray transform of $F$ at $x \in \mathbb{R}^n$ in the direction of $\omega \in \mathbb{S}^{n-1}$ is given by

$$IF(t, x, \omega) = \int_{\mathbb{R}} \omega \cdot F(t, x + s\omega)\mathrm{d}s.$$

Now let $\eta := (\eta_1, \eta_2, \cdots, \eta_n) \in \mathbb{R}^n$ be arbitrary and denote $\omega := (\omega^1, \omega^2, \cdots, \omega^n) \in \mathbb{S}^{n-1}$. Then we have

$$(\eta \cdot \nabla_x) IF(t, x, \omega) = \sum_{i,j=1}^{n} \int_{\mathbb{R}} \omega^i \eta_j \partial_j F_i(t, x + s\omega)\mathrm{d}s. \tag{6.6}$$

Since $F$ has compact support therefore using the Fundamental theorem of calculus, we have

$$\int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}s}(\eta \cdot F)(t, x + s\omega)\mathrm{d}s = 0$$

which gives

$$\sum_{i,j=1}^{n} \int_{\mathbb{R}} \omega^i \eta_j \partial_i F_j(t, x + s\omega)\mathrm{d}s = 0. \tag{6.7}$$

Subtracting (6.7) from (6.6), we get

$$\sum_{i,j=1}^{n} \int_{\mathbb{R}} \omega^i \eta_j h_{ij}(t, x + s\omega)\mathrm{d}s = 0, \text{ for } x \in \mathbb{R}^n \text{ and } \omega \in \mathbb{S}^{n-1} \text{ near a fixed } \omega_0 \in \mathbb{S}^{n-1} \tag{6.8}$$

where $h_{ij}$ is an $n \times n$ matrix with entries

$$h_{ij}(t, x) = \left(\partial_j F_i - \partial_i F_j\right)(t, x), \text{ for } 1 \leq i, j \leq n.$$

Define the Fourier transform of $\sum_{i,j=1}^n \omega^i\eta_j h_{ij}(t,x)$ with respect to space variable $x$ by

$$\sum_{i,j=1}^n \omega^i\eta_j\widehat{h}_{ij}(t,\xi)=\sum_{i,j=1}^n\int_{\mathbb{R}^n}\omega^i\eta_j h_{ij}(t,x)e^{-i\xi\cdot x}\mathrm{d}x,\ \xi\in\mathbb{R}^n.$$

Now decomposing $\mathbb{R}^n=\mathbb{R}\omega\oplus\omega^\perp$ and using (6.8), we get

$$\sum_{i,j=1}^n \omega^i\eta_j\widehat{h}_{ij}(t,\xi)=0,\ \text{for all } \eta\in\mathbb{R}^n,\ \xi\in\omega^\perp \text{ and } \omega \text{ near } \omega_0. \tag{6.9}$$

The goal is to prove that $\widehat{h}_{ij}(t,\xi)=0,$ for $\xi\in\omega^\perp$ with $\omega$ near $\omega_0$ and for each $t\in(0,T)$. From the definition of $\widehat{h}_{ij}(t,\xi)$, it is clear that

$$\widehat{h}_{ii}(t,\xi)=0 \text{ and } \widehat{h}_{ij}(t,\xi)=-\widehat{h}_{ji}(t,\xi),\ \text{for } 1\le i,j\le n.$$

For $n=2$, equation 6.9 gives us

$$(\omega^1\eta_2-\omega^2\eta_1)\widehat{h}_{12}(t,\xi)=0,\ \text{for } \eta\in\mathbb{R}^2,\ \xi\in\omega^\perp \text{ and } \omega \text{ near } \omega_0. \tag{6.10}$$

Now choosing $\eta=(\omega_2,-\omega_1)\in\omega^\perp$ in (6.10), we get $\widehat{h}_{12}(\xi)=0$. Next we show that $\widehat{h}_{ij}(t,\xi)=0$ when $n\ge 3$. Let $\{e_j:1\le j\le n\}$ be the standard basis for $\mathbb{R}^n$ where $e_j$ is is given by

$$e_j:=(0,0,\cdots,0,\underbrace{1}_{j\text{th}},0,\cdots,0)$$

and for simplicity we fix $\omega_0=e_1$. Now let $\xi_0=e_2$ be a fixed vector in $\mathbb{R}^n$. Our first aim is to show that $\widehat{h}_{ij}(\xi_0)=0$, for all $1\le i,j\le n$, then later we will prove that $\widehat{h}_{ij}(t,\xi)=0$ for $1\le i,j\le n$ and $\xi$ near $\xi_0$. Following [44], consider a small perturbation $\omega_0(a)$ of vector $\omega_0=e_1$ by

$$\omega_0(a):=\cos ae_1+\sin ae_k \text{ where } 3\le k\le n.$$

Then we have $\omega_0(a)$ is near $\omega_0$ for $a$ near 0 and $\xi_0\cdot\omega_0(a)=0$. Hence using these choices of $\omega_0(a)$ and $\eta=e_j$ in(6.9), we have

$$\cos a\widehat{h}_{1j}(t,\xi_0)+\sin a\widehat{h}_{kj}(t,\xi_0)=0,\ \text{for } 1\le j\le n,\ \ 3\le k\le n \text{ and } a \text{ near } 0.$$

This gives us

$$\widehat{h}_{1j}(t,\xi_0)=0,\ \widehat{h}_{kj}(t,\xi_0)=0,\ \text{for } 1\le j\le n, \text{and } 3\le k\le n.$$

After using the fact that $\widehat{h}_{ij}=-\widehat{h}_{ji}$ for $1\le i,j\le n$, we get

$$\widehat{h}_{ij}(t,\xi_0)=0,\ \text{for } 1\le i,j\le n.$$

Next we show that $\widehat{h}_{ij}(t,\xi)=0$ for $\xi\in\omega^\perp$ with $\omega$ near $\omega_0$. Using the spherical co-ordinates, we choose $\xi\in\mathbb{S}^{n-1}$ as follows

$$\begin{aligned}
\xi^1&=\sin\phi_1\cos\phi_2\\
\xi^2&=\cos\phi_1\\
\xi^3&=\sin\phi_1\sin\phi_2\cos\phi_3\\
&\ \vdots\\
\xi^{n-1}&=\sin\phi_1\sin\phi_2\cdots\sin\phi_{n-2}\cos\theta\\
\xi^n&=\sin\phi_1\sin\phi_2\cdots\sin\phi_{n-2}\sin\theta.
\end{aligned}$$

Let $A$ be an orthogonal matrix such that $A\xi = e_2$, where $A$ is given by

$$A = \begin{pmatrix} \frac{\partial \xi^1}{\partial \phi_1} & \frac{\partial \xi^2}{\partial \phi_1} & \frac{\partial \xi^3}{\partial \phi_1} & \cdots & \frac{\partial \xi^n}{\partial \phi_1} \\ \xi^1 & \xi^2 & \xi^3 & \cdots & \xi^n \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & a_{k3} & \cdots & a_{kn} \\ a_{k+11} & a_{k+11} & a_{k+13} & \cdots & a_{k+1n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}. \tag{6.11}$$

Now choose

$$\widetilde{\omega} = \left( \frac{\partial \xi^1}{\partial \phi_1}, \frac{\partial \xi^2}{\partial \phi_1}, \frac{\partial \xi^3}{\partial \phi_1}, \cdots, \frac{\partial \xi^n}{\partial \phi_1} \right) \in \mathbb{S}^{n-1}$$

then $\widetilde{\omega}$ is near $\omega_0 = e_1$ when $\phi_i$'s and $\theta$ are close to 0. Next choose $\omega_0(a) = \cos a e_1 + \sin a e_l$ with $l \neq 2$, then $\omega_0(a)$ is close to $e_1$ when $a$ is close to zero. Now define $\omega(a)$ by

$$\omega(a) := A^T \omega_0(a) = \begin{pmatrix} \cos a \cos \phi_1 \cos \phi_2 + a_{l1} \sin a \\ -\cos a \sin \phi_1 + a_{l2} \sin a \\ \vdots \\ \cos a \cos \phi_1 \sin \phi_2 \cdots \sin \phi_{n-2} \cos \theta + a_{ln} \sin a \end{pmatrix}.$$

Then we have $\omega(a)$ is close $\widetilde{\omega}$ for $a$ near 0 and $\tilde{\omega}$ is close to $\omega_0$ when $\phi_i$ and $\theta$ are close to zero. Also we can see that $\omega(a) \cdot \xi = A^T \omega_0(a) \cdot \xi = \omega_0(a) \cdot A\xi = \omega_0(a) \cdot e_2 = 0$, hold because of the choice of $\omega_0(a)$. Hence using these choices of $\omega(a)$ and choosing $\eta = e_j$ in (6.9), we have

$$\cos a \left( \sum_{i=1}^{n} \frac{\partial \xi^i}{\partial \phi_1} \widehat{h}_{ij}(t, \xi) \right) + \sin a \left( \sum_{i=1}^{n} a_{li} \widehat{h}_{ij}(t, \xi) \right) = 0, \text{ for} 1 \leq j \leq n \text{ and } a \text{ near } 0.$$

Now since $\sin a$ and $\cos a$ are linearly independent, therefore we get

$$\begin{aligned} \sum_{i=1}^{n} \frac{\partial \xi^i}{\partial \phi_1} \widehat{h}_{ij}(t, \xi) &= 0, \text{ for all } 1 \leq j \leq n \\ \sum_{i=1}^{n} a_{li} \widehat{h}_{ij}(t, \xi) &= 0, \text{ for all } 1 \leq j \leq n \text{ and } l \neq 2. \end{aligned} \tag{6.12}$$

Above equations can be written as

$$\begin{pmatrix} \frac{\partial \xi^1}{\partial \phi_1} & \frac{\partial \xi^2}{\partial \phi_1} & \frac{\partial \xi^3}{\partial \phi_1} & \cdots & \frac{\partial \xi^n}{\partial \phi_1} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & a_{k3} & \cdots & a_{kn} \\ a_{k+1,1} & a_{k+1,2} & a_{k+1,3} & \cdots & a_{k+1,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}. \end{pmatrix} \begin{pmatrix} \widehat{h}_{1j}(t, \xi) \\ \widehat{h}_{2j}(t, \xi) \\ \widehat{h}_{3j}(t, \xi) \\ \vdots \\ \vdots \\ \vdots \\ \widehat{h}_{nj}(t, \xi) \end{pmatrix} = 0. \tag{6.13}$$

Now let us define matrix $B$ and a n-vector $\mathbf{h}_j$ as follows:

$$B = \begin{pmatrix} \frac{\partial \xi^1}{\partial \phi_1} & \frac{\partial \xi^2}{\partial \phi_1} & \frac{\partial \xi^3}{\partial \phi_1} & \cdots & \frac{\partial \xi^n}{\partial \phi_1} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & a_{k3} & \cdots & a_{kn} \\ a_{k+1,1} & a_{k+1,2} & a_{k+1,3} & \cdots & a_{k+1,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix} \textit{ and } \mathbf{h}_j(t,\xi) = \begin{pmatrix} \widehat{h}_{1j}(t,\xi) \\ \widehat{h}_{2j}(t,\xi) \\ \widehat{h}_{3j}(t,\xi) \\ \vdots \\ \vdots \\ \vdots \\ \widehat{h}_{nj}(t,\xi) \end{pmatrix}.$$

Using these, we have equation (6.13), can be written as

$$B\mathbf{h}_j(t,\xi) = 0, \text{ for } 1 \le j \le n. \tag{6.14}$$

Note that the matrix $B$ is obtained from $A$ by removing the second row and it is $(n-1)\times n$ matrix. From the definition of $A$ it is clear that rank of $A$ is $n$, so the rank of $B$ is $n-1$. *i.e.* there exists at-least one non-zero minor of order $n-1$ of the matrix $B$. Without loss of generality assume $B'$ is non-zero minor of order $n-1$, where $B'$ is given by

$$B' = \begin{pmatrix} \frac{\partial \xi^2}{\partial \phi_1} & \frac{\partial \xi^3}{\partial \phi_1} & \cdots & \frac{\partial \xi^n}{\partial \phi_1} \\ a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k2} & a_{k3} & \cdots & a_{kn} \\ a_{k+1,2} & a_{k+1,3} & \cdots & a_{k+1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}.$$

Now using the fact $\widehat{h}_{11}(t,\xi) = 0$ in (6.14), we have

$$B'\mathbf{h}_1'(t,\xi) = 0 \tag{6.15}$$

where $\mathbf{h}_1' = \left(\widehat{h}_{21}(t,\xi), \widehat{h}_{31}(t,\xi), \cdots, \widehat{h}_{n1}(t,\xi)\right)^T$. Since $B'$ has full rank therefore $\mathbf{h}_1'(t,\xi) = 0$. Also using the fact $\widehat{h}_{ij}(t,\xi) = -\widehat{h}_{ji}(t,\xi)$ and $\mathbf{h}_1(t,\xi) = 0$, we have

$$B'\mathbf{h}_j'(t,\xi) = 0, \text{ for } 2 \le j \le n \tag{6.16}$$

where $\mathbf{h}_j'$ is an $(n-1)$ vector obtained after deleting $jth$ entry from $\mathbf{h}_j$. Now using (6.15) and (6.16) in (6.14), we get

$$\mathbf{h}_j(t,\xi) = 0, \quad \text{for } 1 \le j \le n$$

which gives us

$$\widehat{h}_{ij}(t,\xi) = 0, \text{ for } 1 \le i,j \le n \text{ and } \xi \text{ near } e_2.$$

Since $\widehat{h}_{ij}$ are compactly supported therefore using the Paley-Wiener theorem, we have

$$\widehat{h}_{ij}(t,\xi) = 0, \text{ for } 1 \le i,j \le n \ \xi \in \mathbb{R}^n \text{ and } t \in (0,T).$$

Fourier inversion formula gives us $h_{ij}(t,x) = 0$ for $x \in \mathbb{R}^n$ and for each $t \in (0,T)$. Finally after using the definition of $h_{ij}(t,x)$ and the Poincaré lemma, there exists a $\Phi(t,\cdot) \in C_c^\infty(\mathbb{R}^n)$ such that $F(t,x) = \nabla_x \Phi(t,x)$ for $x \in \Omega$ and for each $t \in (0,T)$. This completes the proof of Lemma 6.1. $\square$

6.1. **Proof of Theorem 2.1.** Using (6.5) and the fact that $A$ is time-independent, we have

$$\int_{\mathbb{R}} \omega \cdot A(x+s\omega)\mathrm{d}s = 0, \text{ for } x \in \mathbb{R}^n \text{ with } |\omega-\omega_0| \leq \epsilon.$$

Hence using Lemma 6.1 in the above equation, there exists $\Phi \in W_0^{2,\infty}(\Omega)$ such that

$$A(x) = \nabla_x \Phi(x),\ x \in \Omega. \tag{6.17}$$

This completes the proof for recovery of convection term $A(x)$. Next we prove the uniqueness for the density coefficients $q_i(t,x)$ for $i=1,2$. Since from (6.17), we have $A^{(2)}(x) - A^{(1)}(x) = \nabla_x \Phi(x)$ for some $\Phi \in W_0^{2,\infty}(\Omega)$. Now if replace the pair $(A^{(1)}, q_1)$ by $(A^{(3)}, q_3)$ where $A^{(3)} = A^{(1)} + \nabla_x \Phi$ and $q_3 = q_1$ then using the fact that $\Phi \in W_0^{2,\infty}(\Omega)$ and Equation (2.2), we get $\Lambda_{A^{(3)},q_3} = \Lambda_{A^{(2)},q_2}$. Now repeating the previous arguments and Lemma 6.1, there exists $\Phi_1 \in W_0^{2,\infty}(\Omega)$ such that

$$A^{(3)}(x) - A^{(2)}(x) = \nabla_x \Phi_1(x)$$

which gives us $A^{(3)}(x) = A^{(2)}(x)$ for $x \in \Omega$. Hence using pairs $(A^{(3)}, q_3)$ and $(A^{(2)}, q_2)$ in (5.9) and the fact that $q_3 = q_1$, we get

$$\int_{Q} q(t,x) u_2(t,x) \overline{v(t,x)} \mathrm{d}x\mathrm{d}t = -\int_{\Sigma\setminus G} \partial_\nu u(t,x) \overline{v(t,x)} \mathrm{d}S_x \mathrm{d}t$$

where $q(t,x) := q_1(t,x) - q_2(t,x)$. Now using the expressions for $u_2$ and $v$ from (4.1) and (4.4) respectively and taking $\lambda \to \infty$, we get

$$\int_{Q} q(t,x) e^{-i(\tau t + x\cdot\xi)} \mathrm{d}x\mathrm{d}t = 0, \text{ for } \tau \in \mathbb{R} \text{ and } \xi \in \omega^{\perp}, \text{ where } \omega \in \mathbb{S}^{n-1} \text{ is near } \omega_0.$$

Since $q \in L^{\infty}(Q)$ is zero outside $Q$ therefore by using the Paley-Wiener theorem we have $q_1(t,x) = q_2(t,x)$ for $(t,x) \in Q$. This completes the proof of Theorem 2.1.

6.2. **Proof of Corollary 2.2.** Using Equation (6.5) and Lemma (6.1), we have for every $t \in (0,T)$ there exists $\Phi(t,\cdot) \in W_0^{2,\infty}(\Omega)$ such that

$$A^{(2)}(t,x) - A^{(1)}(t,x) = \nabla_x \Phi(t,x),\ (t,x) \in Q. \tag{6.18}$$

Now using Equations (2.3) and (6.18), we have

$$\begin{cases} \Delta_x \Phi(t,x) = 0,\ x \in \Omega \text{ and for each } t \in (0,T) \\ \Phi(t,x) = 0,\ x \in \partial\Omega \text{ and for each } t \in (0,T). \end{cases}$$

Using the unique solvability for the above equation, we have $\Phi(t,x) = 0$ for $(t,x) \in Q$. Thus from Equation (6.18), we get $A^{(2)}(t,x) = A^{(1)}(t,x)$ for $(t,x) \in Q$. Using this in (5.9) and repeating the previous arguments, we get $q_1(t,x) = q_2(t,x)$, $(t,x) \in Q$.

## Acknowledgments

We thank the anonymous referees for useful comments which helped us to improve the paper. MV thanks Ibtissem Ben Aïcha and Gen Nakamura for the discussions on this problem. Both the authors are thankful to Venky Krishnan for stimulating discussions and many useful suggestions which helped us to improve the paper. SS was partially supported by Matrics grant MTR/2017/000837. The work of MV was supported by NSAF grant (No. U1930402).

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